# $t$-adic Symmetric Multiple Zeta Values for Indices with Alternating $1$ and $3$, Starting with $1$ and Ending with $3$


Kento Fujita

ARISE analytics, Inc.

(Graduate of Rikkyo University)


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# Today's talk

The slide PDF file is here:

This talk is based on

K.F., "$t$-adic symmetric multiple zeta values for indices with alternating $\mathbf{1}$ and $\mathbf{3}$, starting with $\mathbf{1}$ and ending with $\mathbf{3}$",
Res. number theory **11**, 103 (2025).

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# Agenda

1. **Background**
   - Multiple zeta values (MZVs)
   - Finite MZVs, Symmetric MZVs and the Kaneko–Zagier conjecture
   - $\boldsymbol{p}$-adic FMZVs, $\boldsymbol{t}$-adic SMZVs and the refined Kaneko–Zagier conjecture

2. **Main Result**
   - Previous research
   - Statement of the main result
   - Sketch of the proof
   - Summary and Future Work

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2. **Main Result**
   - Previous research
   - Statement of the main result
   - Sketch of the proof
   - Summary and Future Work

**Today's objects:** For $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{Z}_{\geq 1}^r$ with $k_r>1$, $M\in\mathbb{Z}_{>0}$

$$\zeta(\mathbf{k})=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\in\mathbb{R},\quad \zeta_{<M}(\mathbf{k})=\sum_{0<m_1<\cdots<m_r<M}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\in\mathbb{Q},$$

$$\zeta_{\mathcal{A}}(\mathbf{k})=(\zeta_{<p}(\mathbf{k}) \bmod p)_p\in\mathcal{A}=(\Pi_p\mathbb{Z}/p\mathbb{Z})/(\oplus_p\mathbb{Z}/p\mathbb{Z}),$$

$$\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})=\Big((\zeta_{<p}(\mathbf{k}) \bmod p^n)_p\Big)_n\in\widehat{\mathcal{A}}=\lim_{\leftarrow n}(\Pi_p\mathbb{Z}/p^n\mathbb{Z})/(\oplus_p\mathbb{Z}/p^n\mathbb{Z}),$$

$$\zeta_{\widehat{\mathcal{A}}}^{a}(\mathbf{k})=\Big((\zeta_{<(a+1)p}(\mathbf{k})-\zeta_{<ap+1}(\mathbf{k}) \bmod p^n)_p\Big)_n\in\widehat{\mathcal{A}}$$

$$\zeta_{\mathcal{S}}(\mathbf{k})=\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta^*(k_1,\ldots,k_i)\zeta^*(k_r,\ldots,k_{i+1}) \bmod \zeta(2)\in\overline{\mathcal{Z}}=\mathcal{Z}/\zeta(2)\mathcal{Z},$$

$$\zeta_{\widehat{\mathcal{S}}}(\mathbf{k})=\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta^*(k_1,\ldots,k_i)$$
$$\times\sum_{l_r,\ldots,l_{i+1}\in\mathbb{Z}_{\geq 0}}\left[\prod_{j=i+1}^{r}\binom{k_j+l_j-1}{l_j}\right]\zeta^*(k_r+l_r,\ldots,k_{i+1}+l_{i+1})t^{l_r+\cdots+l_{i+1}} \bmod \zeta(2)\in\overline{\mathcal{Z}}[[t]],$$

$$\zeta_{\widehat{\mathcal{S}}}^{s,t}(\mathbf{k})=\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\sum_{l_1,\ldots,l_i\in\mathbb{Z}_{\geq 0}}\left[\prod_{j=1}^{i}\binom{k_j+l_j-1}{l_j}\right]\zeta^*(k_1+l_1,\ldots,k_i+l_i)(-s)^{l_1+\cdots+l_i}$$
$$\times\sum_{l_r,\ldots,l_{i+1}\in\mathbb{Z}_{\geq 0}}\left[\prod_{j=i+1}^{r}\binom{k_j+l_j-1}{l_j}\right]\zeta^*(k_r+l_r,\ldots,k_{i+1}+l_{i+1})t^{l_r+\cdots+l_{i+1}} \bmod \zeta(2)\in\overline{\mathcal{Z}}[[s,t]],$$

# Multiple zeta values (MZVs)



## Notation:

- $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r$ : index
- $\mathbf{k}$ is admissible $\overset{\text{def}}{\Longleftrightarrow}$ $k_r > 1$ or $\mathbf{k} = \emptyset$
- $\mathrm{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ : weight
- $\mathrm{dep}(\mathbf{k}) = r$ : depth

**Definition (MZVs):** For an admissible index $\mathbf{k} = (k_1, \ldots, k_r)$,

$$\zeta(\mathbf{k}) = \zeta(k_1, \ldots, k_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R} \quad (\zeta(\emptyset) := 1).$$

## Remark:

- $\mathrm{dep}(\mathbf{k}) = 1 \Longrightarrow \zeta(k)$: the Riemann zeta value (RZV).
- $\mathbb{Q}$-linear relations: $\zeta(1,2) = \zeta(3)$, $\zeta(\mathbf{k} * \mathbf{l}) = \zeta(\mathbf{k})\zeta(\mathbf{l})$, $\zeta(\mathbf{k} ⧢ \mathbf{l}) = \zeta(\mathbf{k})\zeta(\mathbf{l}), \ldots$

# Multiple zeta values (MZVs)



- $\mathbb{Q}$-vector space spanned by MZVs:

$$\mathcal{Z}_k := \langle \underbrace{\zeta(\mathbf{k}) \mid \mathrm{wt}(\mathbf{k}) = k}_{2^{k-2}} \rangle_{\mathbb{Q}}, \quad \mathcal{Z} := \sum_{k=0}^{\infty} \mathcal{Z}_k.$$

**Conjecture (Zagier):** $d_0 = 1, d_1 = 0, d_2 = 1, d_n = d_{n-2} + d_{n-3}$ $(n \geq 3)$.

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \stackrel{?}{=} d_k.$$

- **Theorem (Goncharov, Terasoma):** $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.

| weight: $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_k$ | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 |
| $2^{k-2}$ | – | – | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |

**Our motivation: To understand what relations hold among MZVs.**

# Finite MZVs (FMZVs)



- Truncated MZVs: For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ and $M \in \mathbb{Z}_{>0}$,

$$\zeta_{<M}(k_1, \dots, k_r) := \sum_{0<m_1<\cdots<m_r<M} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}.$$

**Definition:**

$$\mathcal{A} := \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right),$$

where $p$ runs over all primes.

- $(a_p)_p = (b_p)_p$ in $\mathcal{A} \Leftrightarrow a_p = b_p$ holds for all but finitely many primes $p$
- For $r \in \mathbb{Q}$, $r_p := \begin{cases} r \bmod p & \text{if } \gcd(\text{den. of } r, p) = 1 \\ 0 & \text{if } \gcd(\text{den. of } r, p) \neq 1 \end{cases}$
- injective map $\mathbb{Q} \ni r \mapsto (r_p)_p \in \mathcal{A}$
  $\Longrightarrow \mathcal{A}$ : $\mathbb{Q}$-algebra.

**Definition (FMZVs):** For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r) := (\zeta_{<p}(k_1, \dots, k_r) \bmod p)_p \in \mathcal{A} \quad (\zeta_{\mathcal{A}}(\emptyset) := 1).$$

# The $\mathbb{Q}$-algebra $\mathcal{Z}_{\mathcal{A}}$



- $\mathbb{Q}$-vector space spanned by FMZVs:

$$\mathcal{Z}_{\mathcal{A},k} := \underbrace{\langle \zeta_{\mathcal{A}}(\mathbf{k}) \mid \mathrm{wt}(\mathbf{k}) = k \rangle}_{2^{k-1}}{}_{\mathbb{Q}}, \quad \mathcal{Z}_{\mathcal{A}} := \sum_{k=0}^{\infty} \mathcal{Z}_{\mathcal{A},k}.$$

**Conjecture (Zagier):** $d_{-3} = 1, d_{-2} = 0, d_{-1} = 0, d_n = d_{n-2} + d_{n-3}$ $(n \geq 0)$.

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} \stackrel{?}{=} d_{k-3}.$$

- **Theorem (Akagi-Hirose-Yasuda, Jarossay):** $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} \leq d_{k-3}$.

| weight: $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_{k-3}$ | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 |
| $2^{k-1}$ | – | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 |

**Our motivation: To understand what relations hold among $\zeta_{\mathcal{A}}(\mathbf{k})$.**

# Relations for FMZVs



**Theorem (Hoffman '15, Zhao '08):** For $k \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\mathcal{A}}(k) = (0)_p.$$

**Theorem (Kaneko–Zagier):** For any indices $\mathbf{k}$ and $\mathbf{l}$,

$$\zeta_{\mathcal{A}}(\mathbf{k} * \mathbf{l}) = \zeta_{\mathcal{A}}(\mathbf{k})\zeta_{\mathcal{A}}(\mathbf{l}).$$

**Theorem (Kaneko–Zagier, Ono '17):** For any indices $\mathbf{k}$ and $\mathbf{l}$,

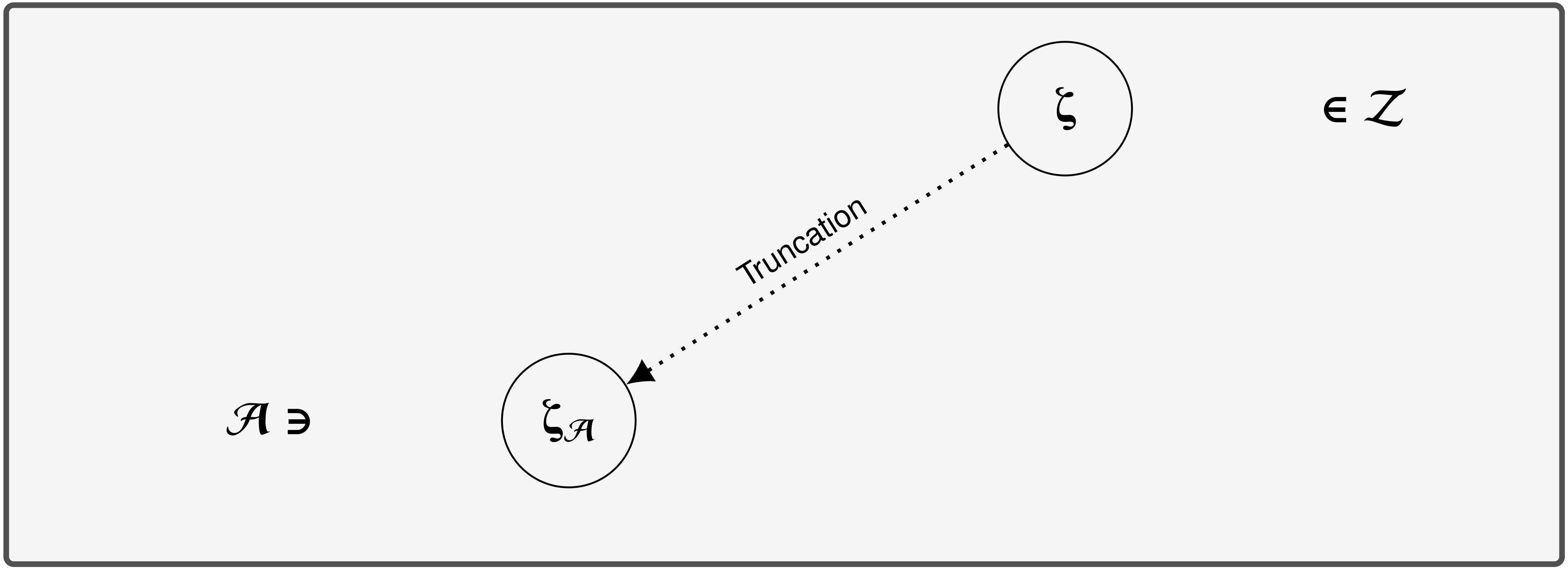
$$\zeta_{\mathcal{A}}(\mathbf{k} ⧢ \mathbf{l}) = (-1)^{\mathrm{wt}(\mathbf{l})}\zeta_{\mathcal{A}}(\mathbf{k}, \overline{\mathbf{l}}),$$

where $\overline{\mathbf{l}} = (l_r, \ldots, l_1)$ for $\mathbf{l} = (l_1, \ldots, l_r)$.

**Attention:**

It has not been shown that $\zeta_{\mathcal{A}}(\mathbf{k}) \neq (0)_p$ for any non-empty index $\mathbf{k}$.

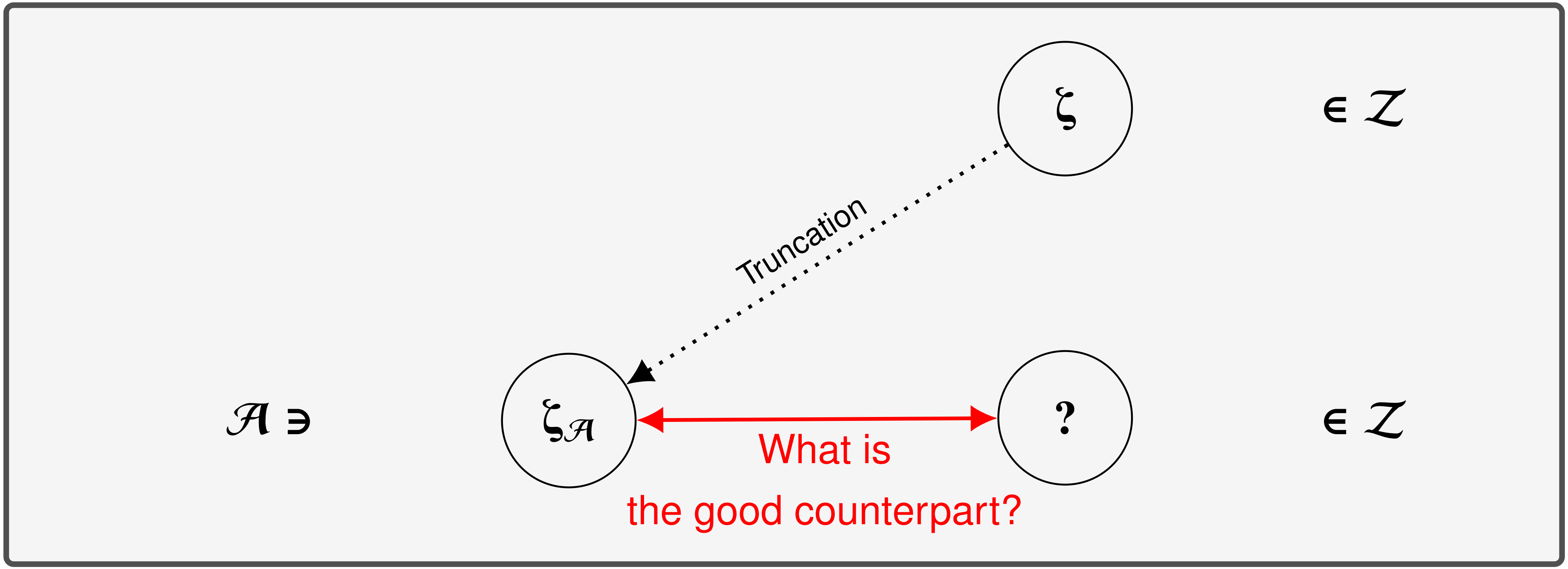
# Summary: $\zeta$ and $\zeta_{\mathcal{A}}$




**Fact:**

- $\zeta_{\mathcal{A}}(2) = (1, 2, 0, 0, \ldots) = (0, 0, 0, 0, \ldots) = (0)_p$.
- $\zeta(2) = \frac{\pi^2}{6}$.

# Summary: $\zeta$ and $\zeta_{\mathcal{A}}$




**Fact:**

- $\zeta_{\mathcal{A}}(2) = (1, 2, 0, 0, \ldots) = (0, 0, 0, 0, \ldots) = (0)_p$.
- $\zeta(2) = \frac{\pi^2}{6}$.

# Symmetric MZVs (SMZVs)



- For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$, $\bullet \in \{*, ш\}$ and
  $\zeta^{\bullet}(k_1, \dots, k_r) := Z^{\bullet}(z_{k_1} \cdots z_{k_r}; T)|_{T=0} \quad (z_k = yx^{k-1} \in \mathbb{Q}\langle x, y\rangle)$,

$$\zeta^{\bullet}_{\mathcal{S}}(k_1, \dots, k_r) := \sum_{i=0}^{r} (-1)^{k_r + \cdots + k_{i+1}} \zeta^{\bullet}(k_1, \dots, k_i) \zeta^{\bullet}(k_r, \dots, k_{i+1}) \in \mathcal{Z},$$

$$\zeta^{\bullet}_{\mathcal{S}}(\emptyset) := 1.$$

**Theorem (Kaneko–Zagier):** For any index **k**,

$$\zeta^{*}_{\mathcal{S}}(\mathbf{k}) - \zeta^{ш}_{\mathcal{S}}(\mathbf{k}) \in \zeta(2)\mathcal{Z}.$$

**Definition (SMZVs):** For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\mathcal{S}}(k_1, \dots, k_r) := \zeta^{\bullet}_{\mathcal{S}}(k_1, \dots, k_r) \bmod \zeta(2) \in \mathcal{Z}/\zeta(2)\mathcal{Z}.$$

**For example:** $\zeta_{\mathcal{S}}(2) = 2\zeta(2) \equiv 0 \bmod \zeta(2)$ (cf. $\zeta_{\mathcal{A}}(2) = (0)_p$).

# Kaneko–Zagier conjecture



- $\mathbb{Q}$-vector space spanned by SMZVs:

$$\mathcal{Z}_{\mathcal{S},k} := \langle \zeta_{\mathcal{S}}(\mathbf{k}) \mid \mathrm{wt}(\mathbf{k}) = k \rangle_{\mathbb{Q}}, \quad \mathcal{Z}_{\mathcal{S}} := \sum_{k=0}^{\infty} \mathcal{Z}_{\mathcal{S},k}.$$

**Theorem (Yasuda '16):** $\mathcal{Z}_{\mathcal{S}} = \mathcal{Z}/\zeta(2)\mathcal{Z}.$

**Kaneko–Zagier conjecture:** There exists a $\mathbb{Q}$-algebra isomorphism

$$\phi_{\mathrm{KZ}} : \mathcal{Z}_{\mathcal{A}} \simeq \mathcal{Z}/\zeta(2)\mathcal{Z}$$

such that $\phi_{\mathrm{KZ}}(\zeta_{\mathcal{A}}(\mathbf{k})) = \zeta_{\mathcal{S}}(\mathbf{k})$ for any index $\mathbf{k}$.

**Remark:**

- **$\zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{S}}(\mathbf{k})$ satisfy exactly the same $\mathbb{Q}$-linear relations !!**

# Relations for $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$



| $\mathcal{A}$ | $\mathcal{S}$ |
|---|---|
| $\zeta_{\mathcal{A}}(k) = (0)_p$ | $\zeta_{\mathcal{S}}(k) = 0$ |
| $\zeta_{\mathcal{A}}(\mathbf{k} * \mathbf{l}) = \zeta_{\mathcal{A}}(\mathbf{k})\zeta_{\mathcal{A}}(\mathbf{l})$ | $\zeta_{\mathcal{S}}(\mathbf{k} * \mathbf{l}) = \zeta_{\mathcal{S}}(\mathbf{k})\zeta_{\mathcal{S}}(\mathbf{l})$ |
| $\zeta_{\mathcal{A}}(\mathbf{k} ш \mathbf{l}) = (-1)^{\mathrm{wt}(\mathbf{l})}\zeta_{\mathcal{A}}(\mathbf{k}, \overline{\mathbf{l}})$ | $\zeta_{\mathcal{S}}(\mathbf{k} ш \mathbf{l}) = (-1)^{\mathrm{wt}(\mathbf{l})}\zeta_{\mathcal{S}}(\mathbf{k}, \overline{\mathbf{l}})$ |

# Problems for $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$



1. **Prove the Kaneko–Zagier conjecture!!**
2. Set $\mathfrak{Z}(k) := \left(\frac{B_{p-k}}{k} \bmod p\right)_p$ ($B_n$: the $n$-th Bernoulli number).

   For odd $k \geq 3$, $\mathfrak{Z}(k) \neq (0)_p$?

   **Remark:** $\mathfrak{Z}(k)_{(p)} \overset{?}{\longleftrightarrow} \zeta(k) \bmod \zeta(2)$:

   $$\zeta(k) \text{ “}\equiv\text{” } \zeta(k-(p-1)) \overset{\text{Euler}}{=} -\frac{B_{p-k}}{p-k} \equiv \mathfrak{Z}(k)_{(p)}$$

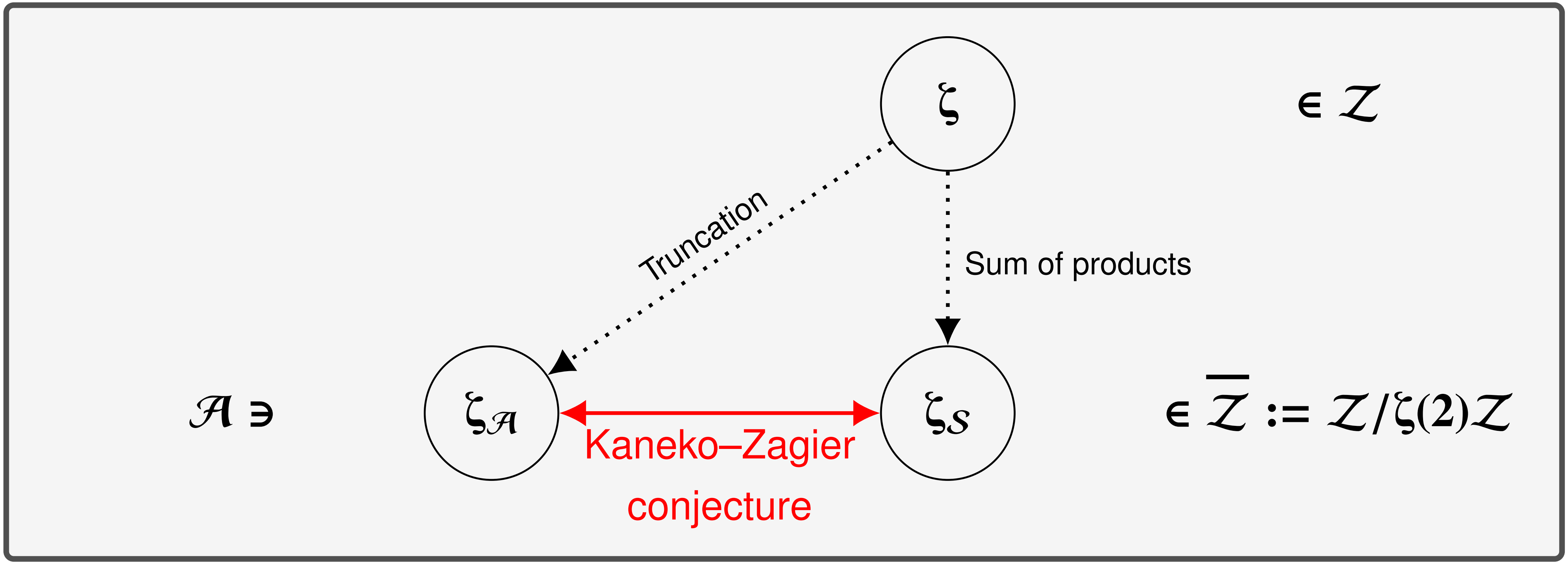
3. What family of relations achieves $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} \leq d_{k-3}$?

   **Remark:**

   $$\zeta_{\mathcal{A}}(\mathbf{k} * (l)) = (0)_p \quad (\forall l \geq 1, \forall \mathbf{k} \text{ s.t. } \mathrm{wt}(\mathbf{k}) + l = k),$$
   $$\zeta_{\mathcal{A}}(\mathbf{k} \,ш\, (\mathbf{l})) = (-1)^{\mathrm{wt}(\mathbf{l})} \zeta_{\mathcal{A}}(\mathbf{k}, \overline{\mathbf{l}}) \quad (\forall \mathbf{k}, \mathbf{l} \text{ s.t. } \mathrm{wt}(\mathbf{k}) + \mathrm{wt}(\mathbf{l}) = k).$$

4. Find and prove the relations that hold for both $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$.
5. Seek common frameworks for $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$. (▶)

# Summary: Kaneko–Zagier conjecture




**Remark:**

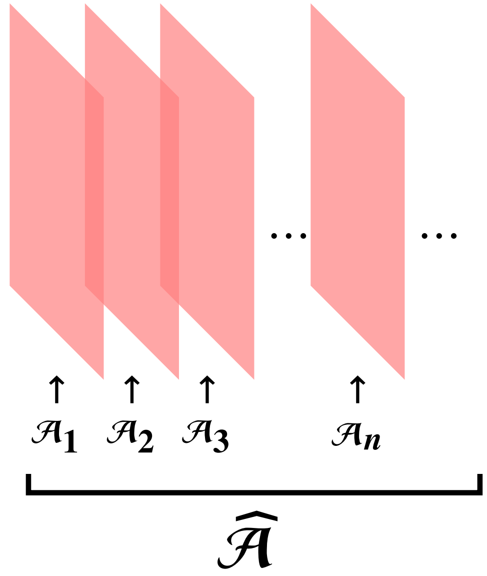
**Kaneko–Zagier conjecture has been further “refined” in recent years!!**

# $p$-adic FMZVs

- Rosen introduced a generalization of $\mathcal{A}$:

$$\mathcal{A}_n = \Big(\prod_p \mathbb{Z}/p^n\mathbb{Z}\Big)\Big/\Big(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}\Big) \quad (\mathcal{A}_1 = \mathcal{A}),$$

$$\widehat{\mathcal{A}} = \varprojlim_n \mathcal{A}_n.$$




- Each $\mathcal{A}_n$ is equipped with the discrete topology.
- $\phi_n : \widehat{\mathcal{A}} \to \mathcal{A}_n \rightsquigarrow \widehat{\mathcal{A}}/\boldsymbol{p}^n\widehat{\mathcal{A}} \cong \mathcal{A}_n$, where $\boldsymbol{p} := ((p \bmod p^n)_p)_n$

**Definition ($p$-adic FMZVs):** For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\widehat{\mathcal{A}}}(k_1, \dots, k_r) := ((\zeta_{<p}(k_1, \dots, k_r) \bmod p^n)_p)_n \in \widehat{\mathcal{A}},$$

$$\zeta_{\mathcal{A}_n}(k_1, \dots, k_r) := (\zeta_{<p}(k_1, \dots, k_r) \bmod p^n)_p \in \mathcal{A}_n.$$

**Remark:** $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\widehat{\mathcal{A}}}(\mathbf{k}) \bmod \boldsymbol{p}\widehat{\mathcal{A}}$: $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})$ is a lift of $\zeta_{\mathcal{A}}(\mathbf{k})$.

# $t$-adic SMZVs



- For $\mathbf{k} = (k_1, \dots, k_r)$: index, $\bullet \in \{*, ш\}$ and $\zeta^\bullet$: $\bullet$-regularized MZVs,

$$\zeta^\bullet_{\widehat{\mathcal{S}}}(\mathbf{k}) := \sum_{i=0}^{r} (-1)^{k_r+\cdots+k_{i+1}} \zeta^\bullet(k_1, \dots, k_i)$$

$$\times \sum_{l_{i+1},\dots,l_r \geq 0} \left[ \prod_{j=i+1}^{r} \binom{k_j+l_j-1}{l_j} \right] \zeta^\bullet(k_r+l_r, \dots, k_{i+1}+l_{i+1}) t^{l_{i+1}+\cdots+l_r} \in \mathcal{Z}[[t]]$$

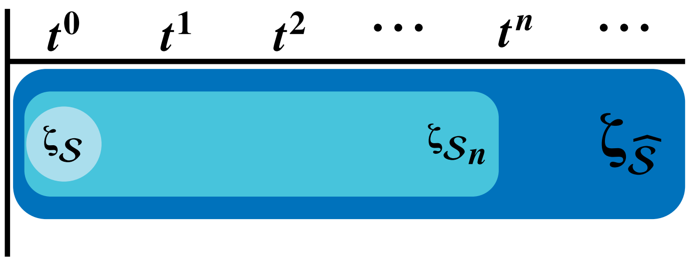
- **Theorem (Jarossay '20, Ono–Seki–Yamamoto '21):**

$$\zeta^*_{\widehat{\mathcal{S}}}(\mathbf{k}) - \zeta^ш_{\widehat{\mathcal{S}}}(\mathbf{k}) \in \zeta(2)\mathcal{Z}[[t]].$$

**Definition ($t$-adic SMZVs):** For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\widehat{\mathcal{S}}}(k_1, \dots, k_r) := \zeta^\bullet_{\widehat{\mathcal{S}}}(k_1, \dots, k_r) \bmod \zeta(2),$$

$$\zeta_{\mathcal{S}_n}(k_1, \dots, k_r) := \zeta_{\widehat{\mathcal{S}}}(k_1, \dots, k_r) \bmod t^n$$




**Remark:** $\zeta_{\mathcal{S}}(\mathbf{k}) = \zeta_{\widehat{\mathcal{S}}}(\mathbf{k}) \bmod t$: $\zeta_{\widehat{\mathcal{S}}}(\mathbf{k})$ is a lift of $\zeta_{\mathcal{S}}(\mathbf{k})$.

# Refined Kaneko–Zagier conjecture



- Let $\mathcal{Z}_{\widehat{\mathcal{A}}}$ and $\mathcal{Z}_{\widehat{\mathcal{S}}}$ be the topological $\mathbb{Q}$-algebras

$$\mathcal{Z}_{\widehat{\mathcal{A}}} = \left\{ \sum_{i \geq 1} a_i \zeta_{\widehat{\mathcal{A}}}(\mathbf{k}_i) p^{n_i} \,\middle|\, \begin{array}{c} a_i \in \mathbb{Q}, \quad \mathbf{k}_i : \text{index}, \\ n_i \in \mathbb{Z}_{\geq 0} \text{ with } n_i \to \infty \ (i \to \infty) \end{array} \right\},$$

$$\mathcal{Z}_{\widehat{\mathcal{S}}} = \left\{ \sum_{i \geq 1} a_i \zeta_{\widehat{\mathcal{S}}}(\mathbf{k}_i) t^{n_i} \,\middle|\, \begin{array}{c} a_i \in \mathbb{Q}, \quad \mathbf{k}_i : \text{index}, \\ n_i \in \mathbb{Z}_{\geq 0} \text{ with } n_i \to \infty \ (i \to \infty) \end{array} \right\}.$$

- **Theorem (Jarossay ’20, Ono-Seki-Yamamoto ’21):**

$\mathcal{Z}_{\widehat{\mathcal{S}}} = (\mathcal{Z}/\zeta(2)\mathcal{Z})[[t]]$.

**Refined Kaneko–Zagier conjecture:** There exists a topological $\mathbb{Q}$-algebra isomorphism

$$\widehat{\phi}_{\mathrm{KZ}} : \mathcal{Z}_{\widehat{\mathcal{A}}} \simeq (\mathcal{Z}/\zeta(2)\mathcal{Z})[[t]]$$

such that

$$\widehat{\phi}_{\mathrm{KZ}}(p) = t, \quad \widehat{\phi}_{\mathrm{KZ}}(\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})) = \zeta_{\widehat{\mathcal{S}}}(\mathbf{k}) \quad (\forall \mathbf{k}\text{: index}).$$

# Relations for $p$-adic FMZVs and $t$-adic SMZVs



- Let $\mathcal{F} \in \{\mathcal{A}, \mathcal{S}\}$ and $\mathfrak{Z}_{\mathcal{F}_n}(k) := \begin{cases} \left(\frac{B_{p^{n-1}(p-1)-k+1}}{k-1+p^{n-1}} \bmod p^n\right) \in \mathcal{A}_n & (\mathcal{F} = \mathcal{A}) \\ \zeta(k) \bmod \zeta(2) \in \overline{\mathcal{Z}}[[t]]/(t^n) & (\mathcal{F} = \mathcal{S}) \end{cases}$

**Theorem ($\mathcal{A}$: Washington'98, Sakugawa-Seki'17, $\mathcal{S}$: trivial by definition):**

For $n, k \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\mathcal{F}_n}(k) = (-1)^k \sum_{l=1}^{n-1} \binom{k+l-1}{l} \mathfrak{Z}_{\mathcal{F}_n}(k+l) x^l_{\mathcal{F}_n}, \quad \left( x_{\mathcal{F}_n} = \begin{cases} (p \bmod p^n)_p & (\mathcal{F} = \mathcal{A}) \\ t \bmod t^n & (\mathcal{F} = \mathcal{S}) \end{cases} \right)$$

**Theorem (Seki '17, Jarossay '20, Ono-Seki-Yamamoto '21):**

For any indices **k** and **l**,

$$\zeta_{\widehat{\mathcal{F}}}(\mathbf{k} * \mathbf{l}) = \zeta_{\widehat{\mathcal{F}}}(\mathbf{k}) \zeta_{\widehat{\mathcal{F}}}(\mathbf{l}),$$

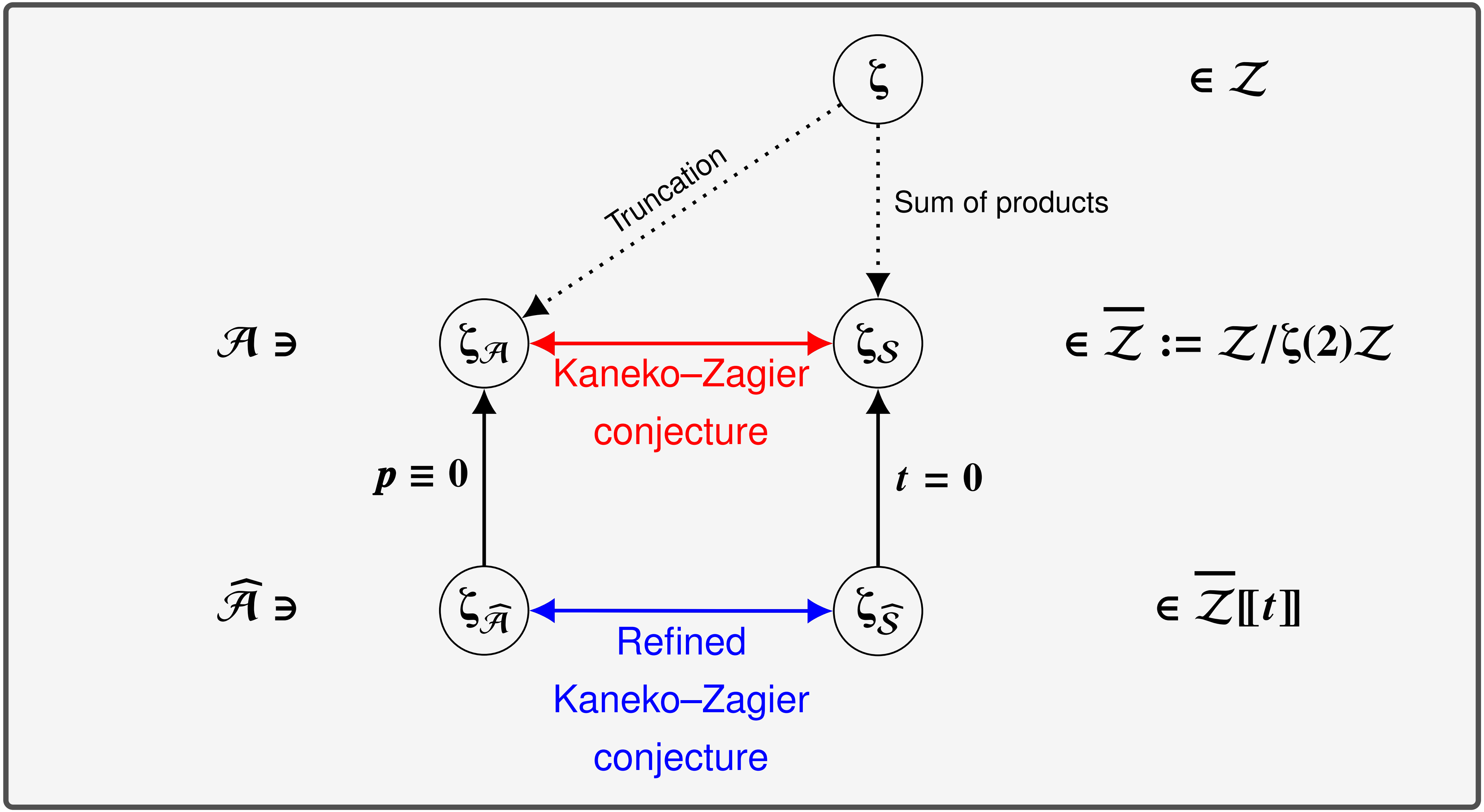
$$\zeta_{\widehat{\mathcal{F}}}(\mathbf{k} ⧢ \mathbf{l}) = (-1)^{\mathrm{wt}(\mathbf{l})} \sum_{\mathbf{m} \in (\mathbb{Z}_{\geq 0})^{\mathrm{dep}(\mathbf{l})}} \prod_{j=1}^{\mathrm{dep}(\mathbf{l})} \binom{l_j + m_j - 1}{m_j} \zeta_{\widehat{\mathcal{F}}}(\mathbf{k}, \overline{\mathbf{l} + \mathbf{m}}) t^{\mathrm{wt}(\mathbf{m})}$$

# Problems: $\zeta_{\widehat{\mathcal{A}}}$ and $\zeta_{\widehat{\mathcal{S}}}$



1. **Prove the refined Kaneko–Zagier conjecture!!**
2. Find and prove the relations that hold for both $\zeta_{\widehat{\mathcal{A}}}$ and $\zeta_{\widehat{\mathcal{S}}}$.
3. Extensions of relation families from $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$ to $\zeta_{\widehat{\mathcal{A}}}$ and $\zeta_{\widehat{\mathcal{S}}}$

# Summary: Refined Kaneko–Zagier conjecture



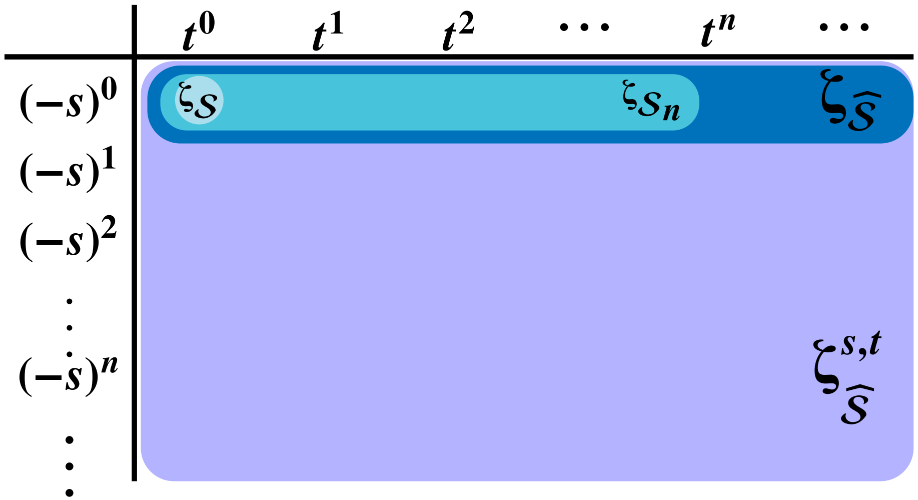
# $(s,t)$-adic SMZVs

**Definition ($(s,t)$-adic SMZVs):** For an index $\mathbf{k}=(k_1,\dots,k_r)$,

$$\zeta_{\widehat{\mathcal{S}}}^{s,t}(\mathbf{k}) := \sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}} \times \sum_{l_1,\dots,l_i\in\mathbb{Z}_{\geq 0}}\left[\prod_{j=1}^{i}\binom{k_j+l_j-1}{l_j}\right]\zeta^*(k_1+l_1,\dots,k_i+l_i)(-s)^{l_1+\cdots+l_i} \times \sum_{l_r,\dots,l_{i+1}\in\mathbb{Z}_{\geq 0}}\left[\prod_{j=i+1}^{r}\binom{k_j+l_j-1}{l_j}\right]\zeta^*(k_r+l_r,\dots,k_{i+1}+l_{i+1})t^{l_r+\cdots+l_{i+1}}$$

$$\bmod \zeta(2) \in \overline{\mathcal{Z}}[[s,t]].$$

- $\zeta_{\widehat{\mathcal{S}}}^{s,t} \overset{s=0}{\rightsquigarrow} \zeta_{\widehat{\mathcal{S}}} \overset{t=0}{\rightsquigarrow} \zeta_{\mathcal{S}}$
- Hirose–Kawamura proved
  - the $(s,t)$-adic harmonic relation
  - the $(s,t)$-adic shuffle relation
  - the $(s,t)$-adic duality
  - the $(s,t)$-adic cyclic sum formula



# The counterpart of $(s,t)$-adic SMZVs

**Definition (Hirose–Kawamura '25):** For $a \in \mathbb{Z}$,

$$\zeta^{a}_{\widehat{\mathcal{A}}}(k_1,\ldots,k_r) := \left(\left(\sum_{ap<m_1<\cdots<m_r<(a+1)p} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \bmod p^n\right)_p\right)_n \in \widehat{\mathcal{A}}$$

**Remark:** $\zeta^{0}_{\widehat{\mathcal{A}}}(\mathbf{k}) = \zeta_{\widehat{\mathcal{A}}}(\mathbf{k})$

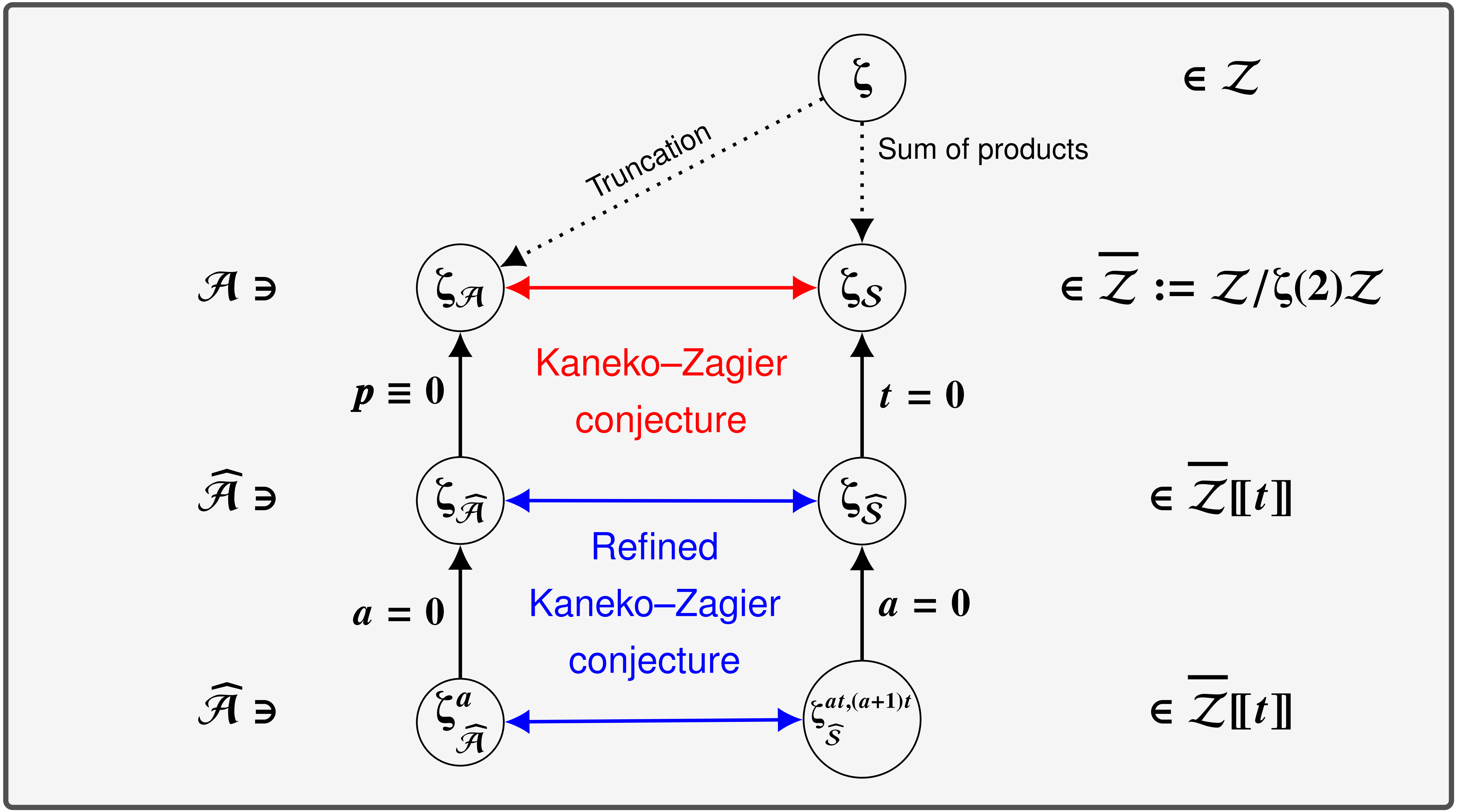
**Theorem (Hirose–Kawamura '25):** Assume that the refined Kaneko–Zagier conjecture is true. Then, $\zeta^{s,t}_{\widehat{\mathcal{S}}}(\mathbf{k})$ is the unique element of $(\mathcal{Z}/\zeta(2)\mathcal{Z})[[s,t]]$ satisfying

$$\phi_{\widehat{\mathrm{KZ}}}\left(\zeta^{at,(a+1)t}_{\widehat{\mathcal{S}}}(\mathbf{k})\right) = \zeta^{a}_{\widehat{\mathcal{A}}}(\mathbf{k})$$

for any integer $a \in \mathbb{Z}$.

# Summary





$\zeta$
$\in \mathcal{Z}$
Truncation
Sum of products
$\mathcal{A} \ni$
$\zeta_{\mathcal{A}}$
$\zeta_{\mathcal{S}}$
$\in \overline{\mathcal{Z}} := \mathcal{Z}/\zeta(2)\mathcal{Z}$
$p \equiv 0$
Kaneko–Zagier
conjecture
$t = 0$
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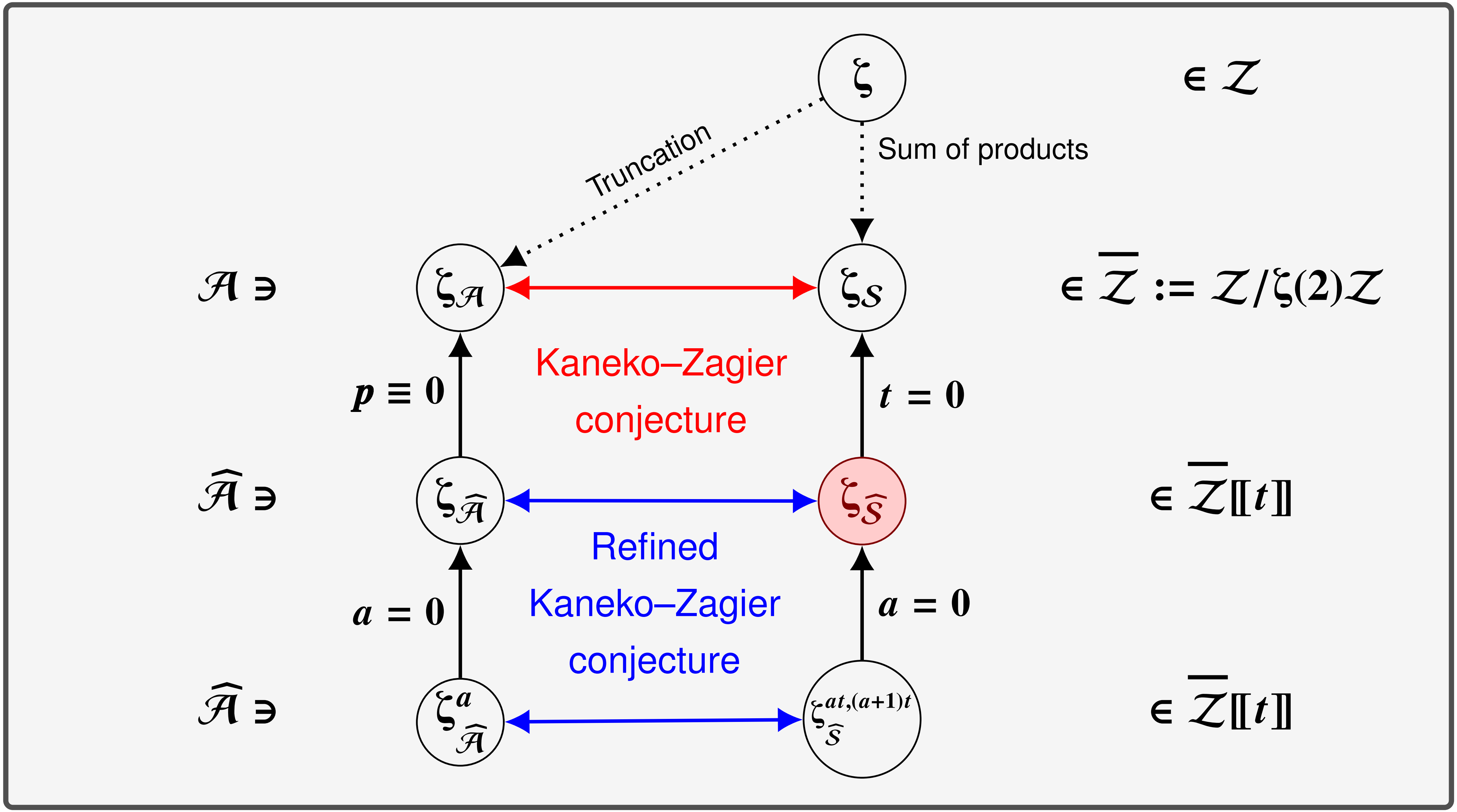
# Summary





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# Agenda

1. **Background**
   - Multiple zeta values (MZVs)
   - Finite MZVs, Symmetric MZVs and the Kaneko–Zagier conjecture
   - $p$-adic FMZVs, $t$-adic SMZVs and the refined Kaneko–Zagier conjecture

2. **Main Result**
   - Previous research
   - Statement of the main result
   - Sketch of the proof
   - Summary and Future Work

# Review of definitions



- For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$, put,

$$\zeta_n^*(k_1, \ldots, k_r) := \sum_{\substack{l_1 \ldots, l_r \in \mathbb{Z}_{\geq 0} \\ l_1 + \cdots + l_r = n}} \left[ \prod_{j=1}^{r} \binom{k_j + l_j - 1}{l_j} \right] \zeta^*(k_1 + l_1, \ldots, k_r + l_r) \in \mathcal{Z}.$$

**Definition (Recall):** For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 1}$,

$$\zeta_{\widehat{\mathcal{S}}}^*(k_1, \ldots, k_r) := \sum_{n=0}^{\infty} \sum_{i=0}^{r} (-1)^{k_r + \cdots + k_{i+1}} \zeta^*(k_1, \ldots, k_i) \zeta_n^*(k_r, \ldots, k_{i+1}) t^n \in \mathcal{Z}[[t]],$$

$$\zeta_{\mathcal{S}_m}^*(k_1, \ldots, k_r) := \zeta_{\widehat{\mathcal{S}}}^*(k_1, \ldots, k_r) \bmod t^m \in \mathcal{Z}[[t]]/(t^m),$$

$$\zeta_{\widehat{\mathcal{S}}}(k_1, \ldots, k_r) := \zeta_{\widehat{\mathcal{S}}}^*(k_1, \ldots, k_r) \bmod \zeta(2) \in \overline{\mathcal{Z}}[[t]].$$

**We will consider special values of $t$-adic SMZVs.**

# Indices in which $1$ and $3$ appear alternately



In this talk, a good index means an index in which $\mathbf{1}$ and $\mathbf{3}$ appear alternately.

**For example:**

$$\emptyset,\ (1,3),\ (3,1,3),\ (\{3,1\}^2) = (3,1,3,1).$$

These good indices are classified as follows:

- Case $(\{1,3\}^a) = (\overbrace{1,3,\dots,1,3}^{a})$
- Case $(\{1,3\}^a,1) = (\overbrace{1,3,\dots,1,3}^{a},1)$
- Case $(\{3,1\}^a) = (\overbrace{3,1,\dots,3,1}^{a})$
- Case $(\{3,1\}^a,3) = (\overbrace{3,1,\dots,3,1}^{a},3)$

Why are these indices "good"?

$\Longrightarrow$ MZVs for these indices can be expressed as polynomials in RZVs.

# MZVs for good indices



**Case $(\{1,3\}^a)$** (conjectured by Zagier '94, proved by Borwein–Bradley–Broadhurst–Lisoněk '01): For $a \in \mathbb{Z}_{\geq 0}$,

$$\zeta(\{1,3\}^a) = \frac{2\pi^{4a}}{(4a+2)!} \in \mathbb{Q}\pi^{4a}.$$

**Case $(\{3,1\}^a, 3)$** (Bowman–Bradley '03): For $a \in \mathbb{Z}_{\geq 0}$,

$$\zeta(\{3,1\}^a, 3) = 4^{-a} \sum_{i=0}^{a} (-1)^i \zeta(4i+3) \zeta(\{4\}^{a-i})$$
$$\in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \zeta(7), \ldots].$$

**Remark:**

$$\zeta(\{4\}^a) = \frac{2^{2a+1}\pi^{4a}}{(4a+2)!} \in \mathbb{Q}\pi^{4a}.$$

# Regularized MZVs for good indices



**Recall:** $\zeta^*$: $*$-regularized MZVs.

**Case $(\{3,1\}^a), (\{1,3\}^a, 1)$** (Bachmann–Charlton '20): For $a \in \mathbb{Z}_{\geq 0}$,

$$\zeta^*(\{3,1\}^a) = 2^{-2a+3} \sum_{\substack{1 \leq i \leq a-1 \\ 0 \leq j \leq a-i-1}} (-1)^{i+1} \zeta^*(4i+1)\zeta(4j+3)\zeta(\{4\}^{a-i-j-1}) + (-1)^a \sum_{i=0}^{a} 4^{-i} \zeta^\star(\{4\}^i)\zeta(\{4\}^{a-i}),$$

$$\zeta^*(\{1,3\}^a, 1) = 2^{-2a+1} \sum_{i=0}^{a} (-1)^i \zeta^*(4i+1)\zeta(\{4\}^{a-i}).$$

**Remark:** The antipode formula gives

$$\sum_{i=0}^{a} (-1)^i \zeta(\{4\}^i)\zeta^\star(\{4\}^{a-i}) = \delta_{a,0}.$$

# Summary: MZVs for good indices

For any good index, $\zeta^*(\mathbf{k})$ is expressed as a polynomial in RZVs.

| $\mathbf{k}$ | $(\{1,3\}^a)$ | $(\{3,1\}^a,3)$ | $(\{3,1\}^a)$ | $(\{1,3\}^a,1)$ |
|---|---|---|---|---|
| $\zeta^*(\mathbf{k})$ | ○ | ○ | ○ | ○ |

## Question:

How about $t$-adic SMZVs for good indices?

## Observation:

In general, $\zeta^*_{\widehat{\mathcal{S}}}(\mathbf{k})$ is not always expressed as a polynomial in RZVs.

$\Longrightarrow$ When does $\zeta^*_{\mathcal{S}_m}(\mathbf{k})$ have a polynomial expression for $m \in \mathbb{Z}_{\geq 0}$?

# Case $(\{3,1\}^a)$



$\mathbf{Coeff}_{t^{m-1}}(\zeta^*_{\widehat{\mathcal{S}}}(\{3,1\}^a))$ is a polynomial in RZVs for $m = 1, 2, 3$.

**Theorem (Hirose–Murahara–Saito '24):** For $a \in \mathbb{Z}_{\geq 0}$

$$\zeta^*_{\mathcal{S}_3}(\{3,1\}^a) = \frac{2(-4)^a}{(4a+2)!}\pi^{4a} + (-1)^{a+1} \sum_{\substack{a_0,a_1 \geq 0 \\ a_0+a_1=2a}} \frac{(-1)^{a_0} 2^{a_0-a_1+2}}{(2a_0+2)!}\pi^{2a_0}\zeta^*(2a_1+1)t$$
$$+ (-1)^a \sum_{\substack{a_0,a_1,a_2 \geq 0 \\ a_0+a_1+a_2=2a}} \frac{(-1)^{a_0} 2^{a_0-a_1-a_2+2}}{(2a_0+2)!}\pi^{2a_0}\zeta^*(2a_1+1)\zeta^*(2a_2+1)t^2$$

**Remark:** For $m \geq 4$, it is reasonable to think that $\zeta^*_{\mathcal{S}_m}(\{3,1\}^a)$ does not have a polynomial expression in RZVs because

$$\mathbf{Coeff}_{t^3}(\zeta^*_{\mathcal{S}_4}(3,1,3,1))$$
$$\equiv \frac{605}{4}\zeta(11) + \frac{19}{4}\zeta(3)^2\zeta(5) + 2\zeta(3)\zeta(3,5) - 2\zeta(3,3,5) \bmod \zeta(2)$$

# Case $(\{1,3\}^a,1)$

$\mathrm{Coeff}_{t^{m-1}}(\zeta^*_{\widehat{\mathcal{S}}}(\{1,3\}^a,1))$ is a polynomial in RZVs for $m=1,2,3$.

**Theorem (Hirose–Murahara–Saito '24):** For $a\in\mathbb{Z}_{\geq 0}$

$$\zeta^*_{\mathcal{S}_3}(\{1,3\}^a,1)=\frac{(-4)^{a+1}}{(4a+4)!}\pi^{4a+2}t + (-1)^a\sum_{\substack{a_0,a_1\geq 0\\ a_0+a_1=2a+1}}\frac{(-1)^{a_1}2^{a_0-a_1+2}}{(2a_0+2)!}\pi^{2a_0}\zeta^*(2a_1+1)t^2$$

**Remark:** For $m\geq 4$, it is reasonable to think that $\zeta^*_{\mathcal{S}_m}(\{1,3\}^a,1)$ does not have a polynomial expression in RZVs because

$$\mathrm{Coeff}_{t^3}(\zeta^*_{\mathcal{S}_4}(1,3,1))\equiv\tfrac{9}{2}\zeta(3)\zeta(5)+\zeta(3,5)\ \mathrm{mod}\ \zeta(2)$$

# Case $(\{3,1\}^a, 3)$



$\mathbf{Coeff}_t(\zeta^*_{\widehat{\mathcal{S}}}(\{3,1\}^a,3))$ is a polynomial in RZVs.

- By the definition of $\zeta_{\widehat{\mathcal{S}}}(\mathbf{k})$, we see that

$$\zeta^*_{\mathcal{S}_1}(\{3,1\}^a,3) = 0$$

**Remark:**

For $m \geq 2$, it is reasonable to think that $\zeta^*_{\mathcal{S}_m}(\{3,1\}^a,3)$ does not have a polynomial expression in RZVs because

$$\mathbf{Coeff}_t(\zeta^*_{\mathcal{S}_2}(3,1,3)) \equiv -5\zeta(3)\zeta(5) - \zeta(3,5) \bmod \zeta(2)$$

# Case ($\{1,3\}^a$) (1/2)

$\mathrm{Coeff}_{t^{m-1}}(\zeta^*_{\widehat{\mathcal{S}}}(\{1,3\}^a))$ is a polynomial in RZVs for $m = 1, 2$.

**Theorem (Hirose–Murahara–Saito '24):** For $a \in \mathbb{Z}_{\geq 0}$

$$\zeta^*_{\mathcal{S}_2}(\{1,3\}^a) = \frac{2(-4)^a}{(4a+2)!}\pi^{4a} + \Bigg( \sum_{\substack{a_0,a_1 \geq 0 \\ a_0+a_1=a}} \frac{(-4)^{a_0+1}(2-(-4)^{-a_1})}{(4a_0+2)!}\pi^{4a_0}\zeta^*(4a_1+1) - (-1)^a \sum_{\substack{a_0,a_1 \geq 0 \\ a_0+a_1=2a \\ a_0,a_1:\text{odd}}} \frac{2^{a_0-a_1+2}}{(2a_0+2)!}\pi^{2a_0}\zeta^*(2a_1+1) \Bigg) t$$

**Remark:** For $m \geq 4$, it is reasonable to think that $\zeta^*_{\mathcal{S}_m}(\{1,3\}^a)$ does not have a polynomial expression in RZVs because

$$\mathrm{Coeff}_{t^3}(\zeta^*_{\mathcal{S}_4}(1,3,1,3)) \equiv -\frac{845}{4}\zeta(11) - \frac{9}{4}\zeta(3)^2\zeta(5) - \zeta(3)\zeta(3,5) + 2\zeta(3,3,5) \bmod \zeta(2)$$

# Case ($\{1,3\}^a$) (2/2)

$\zeta^*_{\mathcal{S}_3}(\{1,3\}^a)$ presumably does not have a polynomial expression in RZVs because

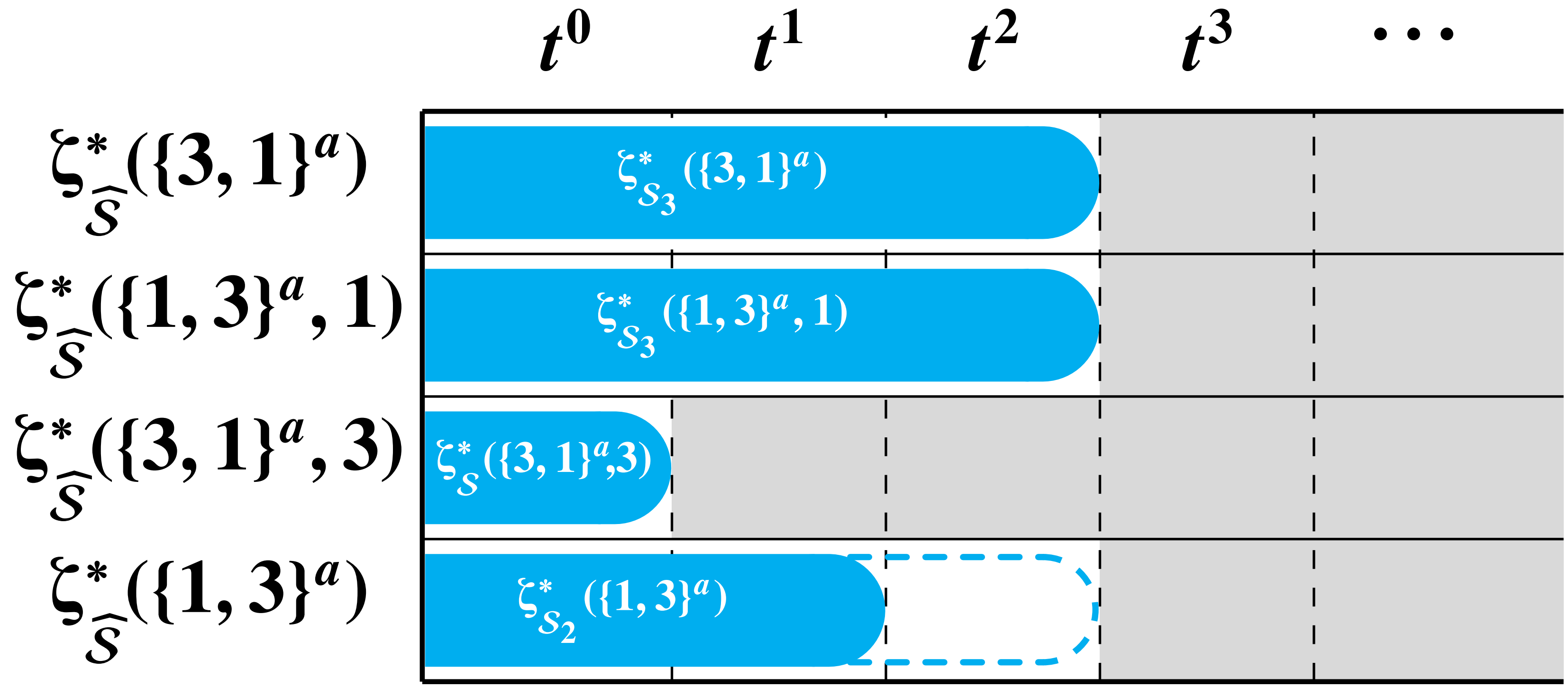
$$\begin{aligned}&\mathbf{Coeff}_{t^2}(\zeta^*_{\mathcal{S}_3}(1,3,1,3))\\ &= \frac{1}{2}\zeta(2)\zeta(3)\zeta(5) + \zeta(2)\zeta(3,5) - \frac{1}{2}\zeta(3)^2\zeta(4) - \frac{1}{4}\zeta(3)\zeta(7) + \frac{81}{5}\zeta(5)^2 - \frac{103}{10}\zeta(10)\end{aligned}$$

However, $\zeta^*_{\mathcal{S}_3}(\{1,3\}^a)$ $\mathbf{mod}$ $\zeta(2)$ is conjectured to be a polynomial in RZVs.

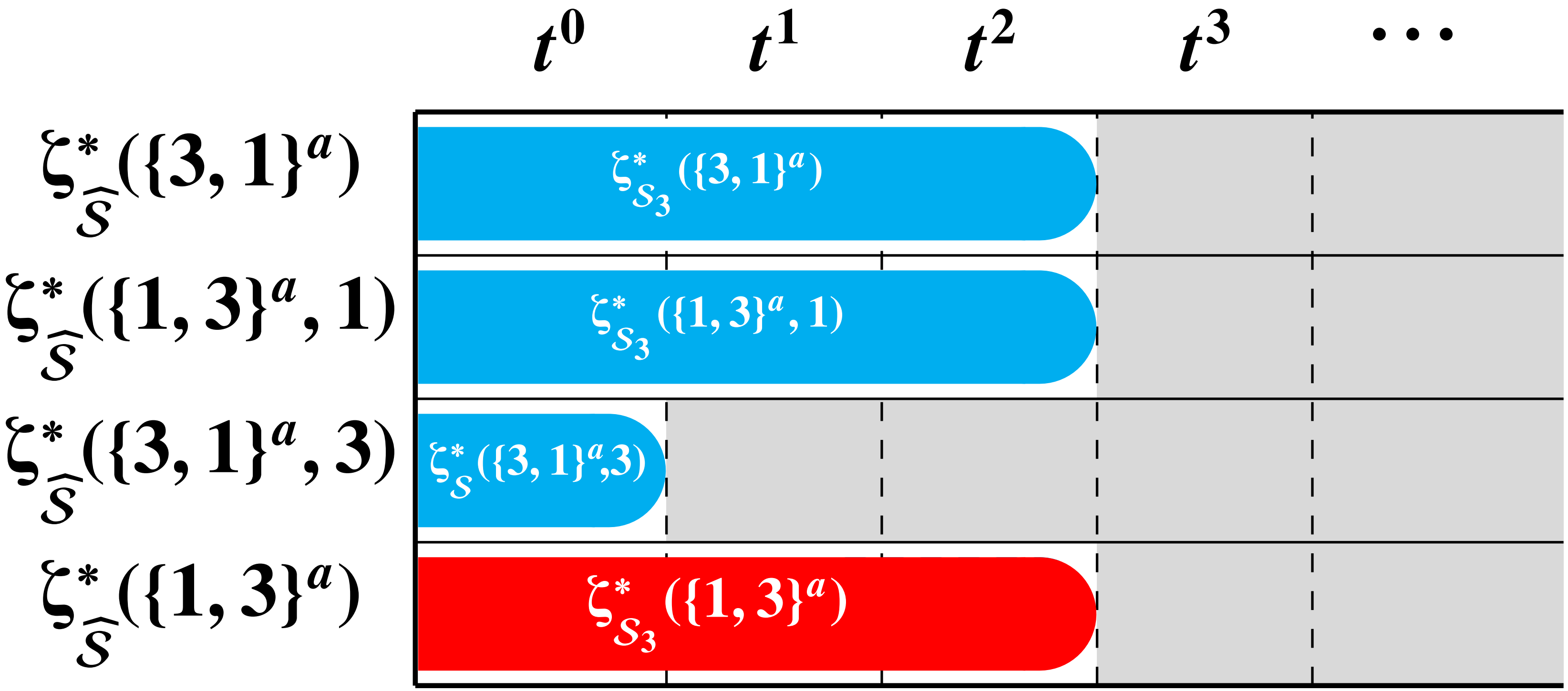
**Conjecture (Hirose–Murahara–Saito ’24):** For $a \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\zeta^*_{\mathcal{S}_3}(\{1,3\}^a) \equiv\ & \delta_{a,0} + 2((-4)^{-a} - 4)\zeta^*(4a+1)t \\ & + \Bigg(-2(-4)^{-a} \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=a-1}} \zeta(4a_1+3)\zeta(4a_2+3) \\ & + 2\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=a}} ((-4)^{-a_1} - 2)((-4)^{-a_2} - 2)\zeta^*(4a_1+1)\zeta^*(4a_2+1)\Bigg)t^2 \\ & \mathbf{mod}\ \zeta(2)\end{aligned}$$

# Summary: $t$-adic SMZVs for good indices



# Summary: $t$-adic SMZVs for good indices




We prove that $\zeta^*_{\mathcal{S}_3}(\{1,3\}^a)$ is
a polynomial in RZVs modulo $\zeta(2)$ !!

# Main result



**Theorem (F., conjectured by Hirose–Murahara–Saito '24):** For $a \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
&\zeta^*_{\mathcal{S}_3}(\{1,3\}^a) \\
&\equiv \delta_{a,0} + 2((-4)^{-a} - 4)\zeta^*(4a+1)t \\
&+ \Bigg(-2(-4)^{-a} \sum_{\substack{a_1,a_2 \geq 0 \\ a_1+a_2=a-1}} \zeta(4a_1+3)\zeta(4a_2+3) \\
&+2 \sum_{\substack{a_1,a_2 \geq 0 \\ a_1+a_2=a}} ((-4)^{-a_1} - 2)((-4)^{-a_2} - 2)\zeta^*(4a_1+1)\zeta^*(4a_2+1)\Bigg)t^2 \bmod \zeta(2)
\end{aligned}$$

## Remark:

- The blue term is congruent to $\zeta^*_{\mathcal{S}_2}(\{1,3\}^a) \bmod \zeta(2)$
- Thus, we explicitly determine the red terms.

# Sketch of HMS's proof (1/2)

**Hirose–Murahara–Saito's idea:** Calculating coefficientwise in $t$.

$$I_n(k_1,\ldots,k_r) := \sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}(k_1,\ldots,k_i) * \sigma_n(k_r,\ldots,k_{i+1}),$$

$$\sigma_n(k_1,\ldots,k_r) := \sum_{\substack{l_1,\ldots,l_r\in\mathbb{Z}_{\geq 0}\\ l_1+\cdots+l_r=n}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right](k_1+l_1,\ldots,k_r+l_r).$$

**Remark:**

- $\zeta_n^*(\mathbf{k}) = \zeta^*(\sigma_n(\mathbf{k}))$
- $\zeta_{\widehat{\mathcal{S}}}^*(\mathbf{k}) = \sum_{n=0}^{\infty}\underbrace{\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta^*(k_1,\ldots,k_i)\zeta_n^*(k_r,\ldots,k_{i+1})}_{=\zeta^*(I_n(k_1,\ldots,k_r))}\, t^n$
- $\zeta_{\mathcal{S}_2}^*(\{1,3\}^a) = \zeta^*(I_0(\{1,3\}^a)) + \zeta^*(I_1(\{1,3\}^a))t$

# Sketch of HMS's proof (2/2)



**Key formula:** $\star : I_0(\{1,3\}^a) = (-1)^a(\{4\}^a)$

- $\zeta^*(I_0(\{1,3\}^a)) \overset{\star}{=} (-1)^a\zeta(\{4\}^a) = \frac{2(-4)^a}{(4a+2)!}\pi^{4a}$

- $\underbrace{\zeta^*(I_1(\{1,3\}^a))}_{\text{Target}} + \underbrace{\zeta^*(I_1(\{3,1\}^a))}_{=\text{ Poly. in RZVs}} = \zeta^*_1(I_0(\{1,3\}^a))$

  $\overset{\star}{=} (-1)^a\zeta^*_1(\{4\}^a)$

  Definition of $\zeta^*_1$, telescoping sum:

  $= \underbrace{4\sum_{i=1}^{a-1}\zeta(4i+5)\zeta(\{4\}^{a-i-1})}_{\text{Poly. in RZVs}}$

$\therefore$ $\left.\begin{array}{l}\zeta^*(I_0(\{1,3\}^a)) \\ \zeta^*(I_1(\{1,3\}^a))\end{array}\right\}$: Poly. in RZVs

$\Longrightarrow$ $\zeta^*_{\mathcal{S}_2}(\{1,3\}^a)$ : Poly. in RZVs

| | $t^0$ | $t^1$ |
|---|---|---|
| $\zeta^*_{\mathcal{S}_2}(\{1,3\}^a)$ | $\zeta^*(I_0(\{1,3\}^a))$ | $\zeta^*(I_1(\{1,3\}^a))$ + |
| $\zeta^*_{\mathcal{S}_2}(\{3,1\}^a)$ | $\zeta^*(I_0(\{3,1\}^a))$ | $\zeta^*(I_1(\{3,1\}^a))$ |

# Sketch of our proof (1/4)

**Our idea:** Generalizing $I_n(\mathbf{k})$

**Definition:** For $m, n \in \mathbb{Z}_{\geq 0}$,

$$ {}_mI_n(k_1,\dots,k_r) = \sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\sigma_m(k_1,\dots,k_i) * \sigma_n(k_r,\dots,k_{i+1}) $$

$$ \left(\text{cf.}\quad I_n(k_1,\dots,k_r) = \sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}(k_1,\dots,k_i) * \sigma_n(k_r,\dots,k_{i+1})\right) $$

**Fact:**

- ${}_0I_n(\mathbf{k}) = I_n(\mathbf{k})$
- ${}_mI_n(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\,{}_nI_m(\overline{\mathbf{k}})$
- $\zeta^*_{\mathcal{S}_3}(\{1,3\}^a) = \underbrace{\zeta^*({}_0I_0(\{1,3\}^a)) + \zeta^*({}_0I_1(\{1,3\}^a))\,t}_{\text{Already determined}} + \underbrace{\zeta^*({}_0I_2(\{1,3\}^a))}_{\text{Target}}\,t^2$

# Sketch of our proof (2/4)



$$\underbrace{\zeta^*({}_0I_2(\{1,3\}^a))}_{\text{Target}} + \underbrace{\zeta^*({}_1I_1(\{1,3\}^a))}_{?} + \underbrace{\zeta^*({}_2I_0(\{1,3\}^a))}_{\heartsuit}$$

$$= \zeta_2^*({}_0I_0(\{1,3\}^a))$$

$$= (-1)^a \zeta_2^*(\{4\}^a)$$

Definition of $\zeta_2^*$

telescoping sum

$$\equiv \underbrace{8 \sum_{\substack{a_1,a_2 \geq 0 \\ a_1+a_2=a}} \zeta^*(4a_1+1)\zeta^*(4a_2+1)}_{\text{Poly. in RZVs}} \bmod \zeta(2)$$

$$\heartsuit : \zeta^*({}_2I_0(\{1,3\}^a)) = \zeta^*({}_0I_2(\{3,1\}^a)) \quad \left(= \mathrm{Coeff}_{t^2}(\zeta^*_{S_3}(\{3,1\}^a))\right)$$

$$\equiv 2(-4)^{-a} \sum_{\substack{a_1,a_2 \geq 0 \\ a_1+a_2=a}} \zeta^*(2a_1+1)\zeta^*(2a_2+1) \bmod \zeta(2)$$

# Sketch of our proof (3/4)



**Proposition (F.):** For $a \in \mathbb{Z}_{\geq 0}$,

$$\zeta^*({}_1I_1(\{1,3\}^a)) \equiv -4 \sum_{\substack{a_1,a_2 \geq 0 \\ a_1+a_2=a}} \left((-4)^{-a} - (-4)^{-a_1} - (-4)^{-a_2}\right) \zeta^*(4a_1+1)\zeta^*(4a_2+1) \bmod \zeta(2)$$

## Outline of the proof:

- Generating functions are used in the proof.
- $\zeta^*({}_1I_1(\{1,3\}^a))$
  $$= \sum_{i=0}^{a} \underbrace{\zeta_1(\{1,3\}^i)}_{\text{Known}} \zeta_1^*(\{3,1\}^{a-i}) - \sum_{i=1}^{a} \underbrace{\zeta_1^*(\{1,3\}^{i-1},1)}_{\text{Known}} \zeta_1^*(\{3,1\}^{a-i},3)$$
- $\zeta_1^*(\{3,1\}^{a+1}) - \sum_{i=0}^{a} 2(-4)^{-i}\zeta^*(4i+1)\zeta_1^*(\{3,1\}^{a-i},3) \equiv 2((-4)^{-a-1}-2)\zeta(4a+5)$

# Sketch of our proof (4/4)

$$\underbrace{\zeta^*({}_0I_2(\{1,3\}^a))}_{\text{Target}} + \underbrace{\zeta^*({}_1I_1(\{1,3\}^a))}_{\text{Poly. in RZVs}} + \underbrace{\zeta^*({}_2I_0(\{1,3\}^a))}_{\text{Poly. in RZVs}}$$

$$\equiv (\text{Poly. in RZVs}) \bmod \zeta(2)$$

$$\Longrightarrow \zeta^*_{\mathcal{S}_3}(\{1,3\}^a) = \underbrace{\zeta^*({}_0I_0(\{1,3\}^a))}_{\text{Poly. in RZVs}} + \underbrace{\zeta^*({}_0I_1(\{1,3\}^a))}_{\text{Poly. in RZVs}}\, t + \underbrace{\zeta^*({}_0I_2(\{1,3\}^a))}_{\text{Poly. in RZVs}}\, t^2$$

This completes the proof. ■

| $\zeta^*({}_mI_n)$ | $n=0$ | $n=1$ | $n=2$ | |
|---|---|---|---|---|
| $m=0$ | $\zeta^*({}_0I_0)$ | $\zeta^*({}_0I_1)$ | $\zeta^*({}_0I_2)$ | $\zeta_{\widehat{\mathcal{S}}}$ |
| $m=1$ | $\zeta^*({}_1I_0)$ | $\zeta^*({}_1I_1)$ | $\zeta^*({}_1I_2)$ | |
| $m=2$ | $\zeta^*({}_2I_0)$ | $\zeta^*({}_2I_1)$ | $\zeta^*({}_2I_2)$ | |

# In terms of $(s,t)$-adic SMZVs

**Recall:**

$$\zeta_{\widehat{\mathcal{S}}}^{s,t}(\mathbf{k}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\underbrace{\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta_m^*(k_1,\ldots,k_i)\zeta_n^*(k_r,\ldots,k_{i+1})}_{\zeta^*({}_mI_n(k_1,\ldots,k_r))}(-s)^m t^n$$

Thus, $\zeta^*({}_1I_1(\{1,3\}^a))$ can be regarded as the coefficient of $-st$ in $\zeta_{\widehat{\mathcal{S}}}^{s,t}(\{1,3\}^a)$:

| $\zeta^*({}_mI_n)$ | $t^0$ | $t^1$ | $t^2$ | |
|---|---|---|---|---|
| $(-s)^0$ | $\zeta^*({}_0I_0)$ | $\zeta^*({}_0I_1)$ | $\zeta^*({}_0I_2)$ | $\zeta_{\widehat{\mathcal{S}}}$ |
| $(-s)^1$ | $\zeta^*({}_1I_0)$ | $\zeta^*({}_1I_1)$ | $\zeta^*({}_1I_2)$ | |
| $(-s)^2$ | $\zeta^*({}_2I_0)$ | $\zeta^*({}_2I_1)$ | $\zeta^*({}_2I_2)$ | $\zeta_{\widehat{\mathcal{S}}}^{s,t}$ |

$\mathrm{Coeff}_{-st}(\zeta_{\widehat{\mathcal{S}}}^{s,t}(\{1,3\}^a))$

$$= -4\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n}}\left((-4)^{-a}-(-4)^{-a_1}-(-4)^{-a_2}\right)\zeta^*(4a_1+1)\zeta^*(4a_2+1)$$

(▶)

# Why is it difficult to compute $\zeta^*_{\mathcal{S}_3}(\{1,3\}^a)$?



- $\zeta^*_{\mathcal{S}_3}(\{3,1\}^a) = \zeta^*({}_0I_0(\{3,1\}^a)) + \zeta^*({}_0I_1(\{3,1\}^a))t + \zeta^*({}_0I_2(\{3,1\}^a))\, t^2$

$\zeta^*({}_0I_2(\{3,1\}^a))$

$$= \sum_{i=0}^{a} \underbrace{\zeta^*(\{3,1\}^i)}_{=\text{Poly. in RZVs}} \underbrace{\zeta^*_2(\{1,3\}^{a-i})}_{=\text{Poly. in RZVs}} - \sum_{i=1}^{a} \underbrace{\zeta(\{3,1\}^{i-1},3)}_{=\text{Poly. in RZVs}} \underbrace{\zeta^*_2(\{1,3\}^{a-i},1)}_{=\text{Poly. in RZVs}}$$

- $\zeta^*_{\mathcal{S}_3}(\{1,3\}^a) = \zeta^*({}_0I_0(\{1,3\}^a)) + \zeta^*({}_0I_1(\{1,3\}^a))t + \zeta^*({}_0I_2(\{1,3\}^a))\, t^2$

$\zeta^*({}_0I_2(\{1,3\}^a))$

$$= \sum_{i=0}^{a} \underbrace{\zeta(\{1,3\}^i)}_{=\text{Poly. in RZVs}} \underbrace{\zeta^*_2(\{3,1\}^{a-i})}_{\stackrel{?}{\equiv}\text{Poly. in RZVs}} - \sum_{i=1}^{a} \underbrace{\zeta^*(\{1,3\}^{i-1},1)}_{=\text{Poly. in RZVs}} \underbrace{\zeta^*_2(\{3,1\}^{a-i},3)}_{\stackrel{?}{\neq}\text{Poly. in RZVs}}$$

(▶)

# Summary



- **The Hirose–Murahara–Saito's conjecture is true:**

$$
\begin{aligned}
\zeta^*_{\mathcal{S}_3}(\{1,3\}^a) \equiv \delta_{a,0} &+ 2((-4)^{-a} - 4)\zeta^*(4a+1)t \\
&+ \Biggl(-2(-4)^{-a} \sum_{\substack{a_1,a_2\geq 0 \\ a_1+a_2=a-1}} \zeta(4a_1+3)\zeta(4a_2+3) \\
&+ 2\sum_{\substack{a_1,a_2\geq 0 \\ a_1+a_2=a}} ((-4)^{-a_1}-2)((-4)^{-a_2}-2)\zeta^*(4a_1+1)\zeta^*(4a_2+1)\Biggr)t^2
\end{aligned}
$$

- The key value for the proof is $\zeta^*({}_1I_1(\{1,3\}^a))$, which is regarded as **the coefficient of $-st$ in $\zeta^{s,t}_{\widehat{\mathcal{S}}}(\{1,3\}^a)$**:

$$
\begin{aligned}
&\zeta^*({}_1I_1(\{1,3\}^a)) \quad (= \mathrm{Coeff}_{-st}(\zeta^{s,t}_{\widehat{\mathcal{S}}}(\{1,3\}^a))) \\
&\equiv -4\sum_{\substack{a_1,a_2\geq 0 \\ a_1+a_2=a}} \left((-4)^{-a} - (-4)^{-a_1} - (-4)^{-a_2}\right)\zeta^*(4a_1+1)\zeta^*(4a_2+1) \bmod \zeta(2)
\end{aligned}
$$

# Future Work



- Do the $\widehat{\mathscr{A}}$-analogues of these relations also have polynomial expressions?
- Is it possible to calculate the value $\zeta^*({}_nI_n(\{1,3\}^a))$ for $n \geq 2$?

| $\zeta^*({}_mI_n)$ | $t^0$ | $t^1$ | $t^2$ | $t^3$ | |
|---|---|---|---|---|---|
| $(-s)^0$ | $\zeta^*({}_0I_0)$ | $\zeta^*({}_0I_1)$ | $\zeta^*({}_0I_2)$ | $\zeta^*({}_0I_3)$ | $\zeta_{\widehat{\mathcal{S}}}$ |
| $(-s)^1$ | $\zeta^*({}_1I_0)$ | $\zeta^*({}_1I_1)$ | $\zeta^*({}_1I_2)$ | $\zeta^*({}_1I_3)$ | |
| $(-s)^2$ | $\zeta^*({}_2I_0)$ | $\zeta^*({}_2I_1)$ | $\zeta^*({}_2I_2)$ | $\zeta^*({}_2I_3)$ | |
| $(-s)^3$ | $\zeta^*({}_3I_0)$ | $\zeta^*({}_3I_1)$ | $\zeta^*({}_3I_2)$ | $\zeta^*({}_3I_3)$ | $\zeta_{\widehat{\mathcal{S}}}^{s,t}$ |

# Thank you very much for your attention!!

# $\zeta^*({}_1I_1(\mathbf{k}))$ for the other good indices.



**Proposition (F.):** Let $n \in \mathbb{Z}_{\geq 0}$ and $\mathbf{k}$ be an index. If $\mathbf{k} = \overline{\mathbf{k}}$ and $\mathrm{wt}(\mathbf{k})$ is odd, then

$$\zeta^*({}_nI_n(\mathbf{k})) = 0.$$

**Proof:**

By definition, ${}_nI_n(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}{}_nI_n(\overline{\mathbf{k}}) = -{}_nI_n(\mathbf{k}) \implies {}_nI_n(\mathbf{k}) = 0.$ ∎

**For example:**

For good indices $\mathbf{k} = (\{1,3\}^a, 1)$ and $(\{3,1\}^a, 3)$,

$$\zeta^*({}_1I_1(\{1,3\}^a, 1)) = 0,$$
$$\zeta^*({}_1I_1(\{3,1\}^a, 3)) = 0.$$

# Generalization of the trio value



**Proposition (F.):** For an index $\mathbf{k} = (k_1, \dots, k_r)$,

$$\sum_{i=0}^{n} {}_iI_{n-i}(\mathbf{k}) = \sigma_n({}_0I_0(\mathbf{k})).$$

**Remark:**

- $\sigma_n(\mathbf{k} * \mathbf{l}) = \sum_{i=0}^{n} \sigma_i(\mathbf{k}) * \sigma_{n-i}(\mathbf{l})$ implies the Proposition.
- $\zeta_n^*({}_0I_0(\{1,3\}^a)) = (-1)^a \zeta_n^*((\{4\}^a)) \leftarrow$: Poly. in RZV $\mathbf{mod}\,\zeta(2)$ for $n \geq 3$ ??

# Why $-s$?



- Kontsevich's order: $(0) \prec 1 \prec 2 \prec \cdots \prec (\infty = -\infty) \prec \cdots \prec -2 \prec -1 \prec (0)$

$$\zeta_{\widehat{S}}^{s,t}(\mathbf{k}) = \sum_{i=0}^{r} (-1)^{k_r+\cdots+k_{i+1}} \left( \sum_{m=0}^{\infty} \zeta_m^*(k_1,\ldots,k_i)(-s)^m \right)\left( \sum_{n=0}^{\infty} \zeta_n^*(k_r,\ldots,k_{i+1}) t^n \right)$$

$$= \sum_{i=0}^{r} \sum_{\substack{0<m_1<\cdots<m_i \\ 0<m_r<\cdots<m_{i+1}}} \frac{(-1)^{k_r+\cdots+k_{i+1}}}{(m_1+s)^{k_r}\cdots(m_i+s)^{k_i}(m_r-t)^{k_r}\cdots(m_{i+1}-t)^{k_{i+1}}}$$

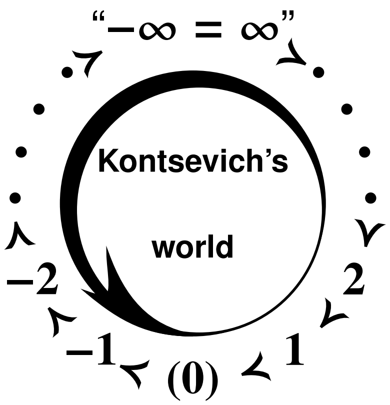
$$= \sum_{i=0}^{r} \sum_{\substack{0<m_1<\cdots<m_i \\ m_{i+1}<\cdots<m_r<0}} \frac{1}{(m_1+s)^{k_r}\cdots(m_i+s)^{k_i}(m_{i+1}+t)^{k_{i+1}}\cdots(m_r+t)^{k_r}}$$

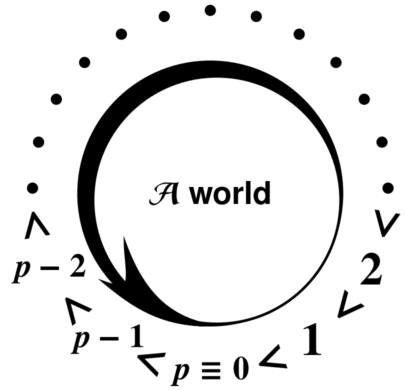
$$= \sum_{\underbrace{m_1 \prec \cdots \prec m_r}_{\text{Kontsevich's order}}} \frac{1}{(m_1+s)^{k_r}\cdots(m_i+s)^{k_i}(m_{i+1}+t)^{k_{i+1}}\cdots(m_r+t)^{k_r}}.$$

# Common frameworks between $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$

- Kontsevich's order: $(0) \prec 1 \prec 2 \prec \cdots \prec (\infty = -\infty) \prec \cdots \prec -2 \prec -1 \prec (0)$

$$\zeta^*_{\mathcal{S}}(\mathbf{k}) = \lim_{M\to\infty} \sum_{\substack{0\prec m_1\prec\cdots\prec m_r(\prec 0)\\ |m_1|,\ldots,|m_r|<M}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$




- BTT philosophy (Bachmann–Takeyama–Tasaka '18):

$$\zeta_{\mathcal{S}}(\mathbf{k}) \xleftarrow[\text{analytic sense}]{q\to 1} q\text{-MZVs} \xrightarrow[\text{algebraic sense}]{q\to 1} \zeta_{\mathcal{A}}(\mathbf{k}).$$

- Unified Multiple Zeta Function (Komori '21):

$$\zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;t_1,t_2) := \sum_{i=0}^{r}(-1)^{s_r+\cdots+s_{i+1}}\zeta(s_1,\ldots,s_i;t_1)\zeta(s_r,\ldots,s_{i+1};t_2),$$

(◀)

# Unified Multiple Zeta Function

**Definition (Komori '21):** For $s_1, \ldots, s_r \in \mathbb{C}$ with sufficiently large $\Re s_1, \ldots, \Re s_r$ and $\Re t < 1$,

$$\zeta_{\widehat{\mathcal{U}}}(s_1, \ldots, s_r; t_1, t_2) := \sum_{i=0}^{r} (-1)^{s_r + \cdots + s_{i+1}} \zeta(s_1, \ldots, s_i; t_1) \zeta(s_r, \ldots, s_{i+1}; t_2),$$

$$\left( \zeta(s_1, \ldots, s_r; t) := \sum_{0 < m_1 < \cdots < m_r} \frac{1}{(m_1 - t)^{s_1} \cdots (m_r - t)^{s_r}}, \quad (-1)^s = e^{\pi i s} \right).$$

**Remark:**

- $\zeta_{\widehat{\mathcal{U}}}(s_1, \ldots, s_r; t_1; t_2)$ is holomorphic for $s_1, \ldots, s_r \in \mathbb{C}$ and $t_1, t_2 \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$.
- $\zeta_{\widehat{\mathcal{U}}}$ unified $\zeta, \zeta_{<M}, \zeta_{\mathcal{A}}, \zeta_{\mathcal{S}}, \zeta_{\widehat{\mathcal{A}}}, \zeta_{\widehat{\mathcal{S}}}, \zeta_{\widehat{\mathcal{S}}}^{s,t}$ by specializing $t_1, t_2$ appropriately.
- $\zeta^*({}_1I_1(\{1,3\}^a))$ is regarded as **the coefficient of $t_1 t_2$ in $\zeta_{\widehat{\mathcal{U}}}(\{1,3\}^a)$** (◀)

# The meaning of $t_1, t_2$

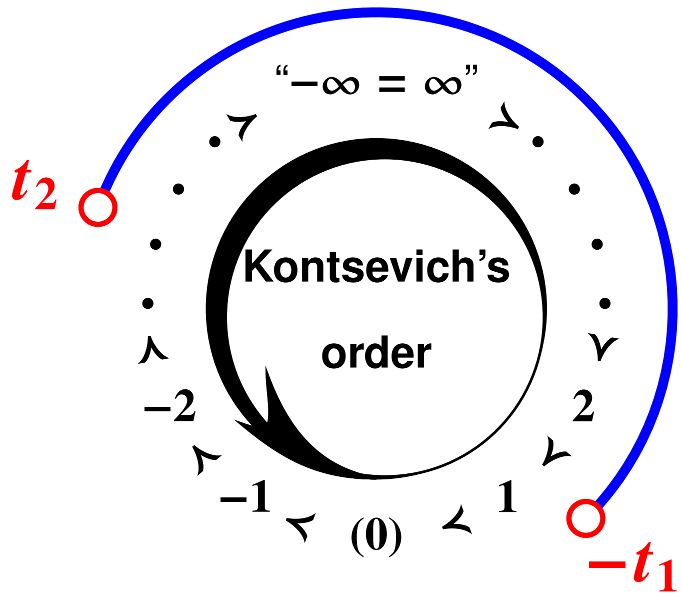
- Kontsevich's order: $(0) \prec 1 \prec 2 \prec \cdots \prec (\infty = -\infty) \prec \cdots \prec -2 \prec -1 \prec (0)$

$$\zeta_{\widehat{\mathcal{U}}}(s_1, \ldots, s_r; t_1, t_2)$$

$$= \sum_{i=0}^{r} (-1)^{s_r+\cdots+s_{i+1}} \zeta(s_1, \ldots, s_i; t_1)\zeta(s_r, \ldots, s_{i+1}; t_2)$$

$$= \sum_{i=0}^{r} \sum_{\substack{0<m_1<\cdots<m_i \\ m_{i+1}<\cdots<m_r<0}} \frac{1}{(m_1-t_1)^{s_r} \cdots (m_i-t_1)^{s_i}(m_{i+1}+t_2)^{s_{i+1}} \cdots (m_r+t_2)^{s_r}}$$

$$\text{“}=\text{”} \sum_{-t_1 \prec m_1 \prec \cdots \prec m_r \prec t_2} \frac{1}{m_1^{s_1} \cdots m_r^{s_r}}$$




- $\zeta_{\widehat{\mathcal{S}}}^{s,t}(k_1, \ldots, k_r) = \zeta_{\widehat{\mathcal{U}}}(k_1, \ldots, k_r; -s, t) \bmod \zeta(2)$

# The values of $\zeta^*(\{3,1\}^a)$ and $\zeta_1^*(\{3,1\}^a)$

**Proposition (K. recall)** For $a \in \mathbb{Z}_{\geq 0}$,

$$\zeta_1^*(\{3,1\}^{a+1}) \equiv \sum_{i=1}^{a+1} 2(-4)^{-i}\zeta(4i+1)\zeta_1^*(\{3,1\}^{a-i},3) + 2((-4)^{-a-1}-2)\zeta(4a+5) \bmod \zeta(2).$$

**Theorem (Bachmann–Charlton '20):** For $a \in \mathbb{Z}_{\geq 0}$,

$$\zeta^*(\{3,1\}^{a+1}) = \sum_{i=1}^{a+1} 2(-4)^{-i}\zeta(4i+1)\zeta(\{3,1\}^{a-i},3) + (-1)^{a+1}\sum_{j=0}^{a+1}(-4)^{-j}\zeta^\star(\{4\}^j)\zeta(\{4\}^{a+1-j}).$$

**Problem:** It is a problem whether we can prove these results in a unified way.

# The value of $\zeta_2^*(\{3,1\}^a)$

$$\zeta_2^*(\{3,1\}^1) = \frac{-19\pi^6}{3780} + \frac{1}{2}\zeta(3)^2,$$

$$\zeta_2^*(\{3,1\}^2) = \frac{-1439\pi^{10}}{14968800} - \frac{\pi^4}{144}\zeta(3)^2 + \frac{\pi^2}{12}\zeta(3)\zeta(5) + \frac{57}{8}\zeta(5)^2 - \frac{1}{4}\zeta(3)\zeta(7) + \frac{\pi^2}{6}\zeta(3,5),$$

$$\zeta_2^*(\{3,1\}^3) = \frac{-7009\pi^{14}}{8817984000} + \frac{17\pi^8}{725760}\zeta(3)^2 - \frac{13\pi^6}{30240}\zeta(3)\zeta(5) - \frac{1}{12}\zeta(3)^3\zeta(5) - \frac{71\pi^4}{960}\zeta(5)^2 + \frac{\pi^4}{288}\zeta(3)\zeta(7) + \frac{47\pi^2}{144}\zeta(5)\zeta(7) + \frac{1}{32}\zeta(7)^2 - \frac{13\pi^6}{15120}\zeta(3,5) - \frac{\pi^2}{48}\zeta(3)\zeta(9) + \frac{541}{48}\zeta(5)\zeta(9) + \frac{1}{16}\zeta(11)\zeta(3) + \frac{5\pi^2}{216}\zeta(3,9).$$

# The value of $\zeta_2^*(\{3,1\}^a,3)$

$$\zeta_2^*(\{3,1\}^1,3) = \frac{1}{6}\zeta(3)^3 - \frac{2\pi^4}{15}\zeta(5) + \frac{83}{6}\zeta(9),$$

$$\zeta_2^*(\{3,1\}^2,3) = \frac{\pi^4}{2160}\zeta(3)^3 + \frac{11\pi^8}{60480}\zeta(5) + \frac{1}{8}\zeta(3)\zeta(5)^2 - \frac{1}{8}\zeta(3)^2\zeta(7) + \frac{101\pi^4}{540}\zeta(9) + \frac{275\pi^2}{12}\zeta(11) + \zeta(3,5,5) - \frac{1971}{8}\zeta(13) + \frac{1}{2}\zeta(3,5)\zeta(5),$$

$$\zeta_2^*(\{3,1\}^3,3) = \frac{\pi^8}{10886400}\zeta(3)^3 - \frac{2812039\pi^{12}}{490377888000}\zeta(5) + \frac{\pi^4}{2880}\zeta(3)\zeta(5)^2 - \frac{\pi^{10}}{748440}\zeta(7) - \frac{\pi^4}{2880}\zeta(3)^2\zeta(7) + \frac{103}{32}\zeta(5)^2\zeta(7) + \frac{1}{32}\zeta(3)\zeta(7)^2 + \frac{\pi^4}{720}\zeta(3,5)\zeta(5) + \frac{9581\pi^8}{10886400}\zeta(9) - \frac{1}{16}\zeta(3)\zeta(5)\zeta(9) - \frac{1}{8}\zeta(3,5)\zeta(9) + \frac{341\pi^6}{6048}\zeta(11) + \frac{1}{32}\zeta(11)\zeta(3)^2 + \frac{5}{24}\zeta(3,9)\zeta(5) + \frac{\pi^4}{360}\zeta(3,5,5) - \frac{893\pi^4}{1440}\zeta(13) + \frac{715\pi^2}{24}\zeta(15) + \frac{1}{4}\zeta(5,3,9) - \frac{1}{4}\zeta(5,5,7) - \frac{9411}{32}\zeta(17).$$