2021/3/9

Weighted sum formulas for symmetric multiple zeta values

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Several multiple zeta values (MZVs) ∼series∼

MZVs and Multi-polylogarithm (MPL)

$$\zeta(k_1,\ldots,k_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathcal{Z} \quad (k_r > 1),$$

$$\zeta^{ш}(k_1,\ldots,k_r) := Z^{ш}(e_{k_1}\cdots e_{k_r};T)|_{T=0} \quad (e_{k_i} := e_1 e_0^{k_i-1} \in \mathbb{Q}\langle e_0,e_1\rangle),$$

$$\mathrm{Li}_{k_1,\ldots,k_r}(X) := \sum_{0<m_1<\cdots<m_r} \frac{X^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}} \quad (0<X<1).$$

Finite MZV (FMZV)

$$\mathcal{A} := \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right)\Big/\left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right)$$

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r) := \left(\zeta_{p-1}(k_1,\ldots,k_r) \bmod p\right)_p \in \mathcal{A}$$

Symmetric MZV (SMZV)

$$\zeta_{\mathcal{S}}^{ш}(\mathbf{k}) := \sum_{i=0}^{r} (-1)^{k_r+\cdots+k_{i+1}} \zeta^{ш}(k_1,\ldots,k_i) \times \zeta^{ш}(k_r,\ldots,k_{i+1})$$

$$\Longrightarrow \zeta_{\mathcal{S}}(\mathbf{k}) := \zeta_{\mathcal{S}}^{ш}(\mathbf{k}) \bmod \zeta(2)\mathcal{Z} \in \mathcal{Z}/\zeta(2)\mathcal{Z}.$$

$$\mathcal{Z}_{\mathcal{A}} := \langle \text{all } \zeta_{\mathcal{A}}(\mathbf{k})\rangle_{\mathbb{Q}} \subset \mathcal{A},$$

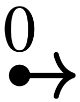
$$\mathcal{Z}_{\mathcal{S}} := \langle \text{all } \zeta_{\mathcal{S}}(\mathbf{k})\rangle_{\mathbb{Q}} = \mathcal{Z}/\zeta(2)\mathcal{Z}.$$

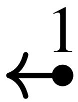
Conjecture (Kaneko–Zagier).

$$\mathcal{Z}_{\mathcal{A}} \overset{?}{\simeq} \mathcal{Z}/\zeta(2)\mathcal{Z} : \zeta_{\mathcal{A}}(\mathbf{k}) \leftrightarrow \zeta_{\mathcal{S}}(\mathbf{k}).$$

Iterated integral

In this talk, we use **Tangential base points**

$0' := 1_0$ 

$1' := (-1)_1$ 

Definition (Regularized iterated integrals)**.** Fix $a_1, \ldots, a_k \in \{0,1\}$ and tangential base points $x, y \in \{0', 1'\}$. For $\gamma : [0,1] \to \mathbb{C}$: path from x to y such that $\gamma((0,1)) \subset \mathbb{C} \setminus \{0,1\}$,

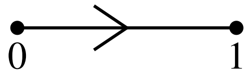
$$\int_{\epsilon < t_1 < \cdots < t_k < 1-\epsilon} \prod_{i=1}^{k} \frac{d\gamma(t_i)}{\gamma(t_i) - a_i} = c_0 + \sum_{j=1}^{k} c_j (\log \epsilon)^j + O(\epsilon \log^{k+1} \epsilon).$$

We define $I_\gamma(x; a_1, \ldots, a_k; y) := c_0$.

Remark. Fix $a_0, a_{k+1} \in \{0,1\}$ with $a_0 \neq a_1$, $a_k \neq a_{k+1}$. For a path γ from a_0 to a_{k+1},

$$I_\gamma(a_0; a_1, \ldots, a_k; a_{k+1}) = \int_{0 < t_1 < \cdots < t_k < 1} \prod_{i=1}^{k} \frac{d\gamma(t_i)}{\gamma(t_i) - a_i}.$$

Several MZVs ∼integrals∼

Let dch be a straight path from 0 to 1: 

: $\mathrm{dch}(t) := t \quad (t \in [0,1])$

(dch: droit chemin = straight path).

MZVs and MPL

$$(-1)^r \zeta(k_1,\ldots,k_r) = I_{\mathrm{dch}}(0;1,0^{k_1-1},\ldots,1,0^{k_r-1};1) \in \mathcal{Z} \quad (k_r > 1),$$

$$(-1)^r \zeta^{ш}(k_1,\ldots,k_r) = I_{\mathrm{dch}}(0';1,0^{k_1-1},\ldots,1,0^{k_r-1};1') \in \mathcal{Z},$$

$$(-1)^r \mathrm{Li}_{k_1,\ldots,k_r}(X) = I_{\mathrm{dch}}(0;1,0^{k_1-1},\ldots,1,0^{k_r-1};X) \quad (0 < X < 1).$$

FMZV

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r) = \left(\mathrm{Coeff}_p\left(I_{\mathrm{dch}}(0;1,0^{k_1-1},\ldots,1,0^{k_r-1};X)\frac{X}{1-X}\right)\right)_p$$

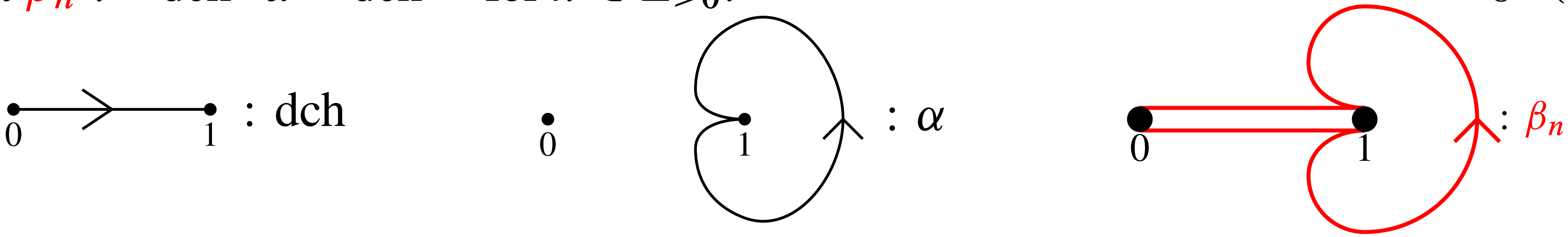
SMZV

What is $\zeta_{\mathcal{S}}(\mathbf{k})$ in terms of iterated integral?

⇒ Refined SMZV!

Refined SMZV (RSMZV)

Set $\beta_n := \mathrm{dch}\cdot\alpha^n\cdot\mathrm{dch}^{-1}$ for $n\in\mathbb{Z}_{>0}$.



For $\mathbf{k}=(k_1,\dots,k_r)$, define a linear map $L_n:\mathbb{Q}\langle e_0,e_1\rangle\to\mathbb{C}$ by

$$L_n(e_{k_1}\cdots e_{k_r}e_1):=I_{\beta_n}(0';1,0^{k_1-1},\dots,1,0^{k_r-1},1;0')$$

$$=\sum_{\substack{0\le a\le b\le r\\ k_j=1\ (a<\forall j\le b)}}\frac{(-2\pi i n)^{b-a+1}}{(b-a+1)!}(-1)^{r+k_r+\cdots+k_{b+1}}\zeta^{ш}(k_1,\dots,k_a)\zeta^{ш}(k_r,\dots,k_{b+1})\in 2\pi i n\mathcal{Z}[2\pi i n].$$

For $w\in\mathbb{Q}\langle e_0,e_1\rangle$, $\exists! L(w;T)\in T\mathcal{Z}[T]$ s.t. $L(w;2\pi i n)=L_n(w)$.

Definition. (Hirose) $\qquad \zeta_{RS}(k_1,\dots,k_r;T):=\dfrac{(-1)^r}{T}L(e_{k_1}\cdots e_{k_r}e_1;T)$

Remark. $\quad \zeta_{RS}(\mathbf{k};0)=\zeta_{S}^{ш}(\mathbf{k}):=\sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta^{ш}(k_1,\dots,k_i)\zeta^{ш}(k_r+\cdots+k_{i+1})$.

Hirose call $\zeta_{RS}(\mathbf{k};2\pi i)$ RSMZVs and prove $\zeta_{RS}(\mathbf{k};2\pi i)\equiv\zeta_{S}^{ш}(\mathbf{k}) \pmod{2\pi i\mathcal{Z}[2\pi i]}$.

Summary

$$(-1)^r\zeta(k_1,\ldots,k_r) = I_{\mathrm{dch}}(0;1,0^{k_1-1},\ldots,1,0^{k_r-1};1) \quad (k_r>1),$$

$$(-1)^r\zeta^{ш}(k_1,\ldots,k_r) = I_{\mathrm{dch}}(0';1,0^{k_1-1},\ldots,1,0^{k_r-1};1'),$$

$$\mathrm{Li}_{k_1,\ldots,k_r}(X) = I_{\mathrm{dch}}(0;1,0^{k_1-1},\ldots,1,0^{k_r-1};X) \quad (0<X<1),$$

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r) = \left(\mathrm{Coeff}_p\left(I_{\mathrm{dch}}(0;1,0^{k_1-1},\ldots,1,0^{k_r-1};X)\frac{X}{1-X}\right)\right)_p,$$

$$\zeta_{RS}(k_1,\ldots,k_r;T) := \frac{(-1)^r}{T}L(e_{k_1}\cdots e_{k_r}e_1;T),$$

$$\zeta_{\mathcal{S}}^{ш}(k_1,\ldots,k_r) = \zeta_{RS}(k_1,\ldots,k_r;0)$$

$$\zeta_{RS}(k_1,\ldots,k_r;2\pi i) := \frac{(-1)^r}{2\pi i}I_\beta(0';1,0^{k_1-1},\ldots,1,0^{k_r-1},1;0')$$

$$\zeta_{\mathcal{S}}(k_1,\ldots,k_r) = \zeta_{RS}(k_1,\ldots,k_r;2\pi i) \bmod 2\pi i\,\mathcal{Z}[2\pi i].$$

Weighted sum formulas for $\zeta_{\mathcal{F}}\quad(\mathcal{F}\in\{\mathcal{A},\mathcal{S}\})$

WSFs for $\zeta_{\mathcal{F}}$ [$\mathcal{A}$: Kamano(2018), $\mathcal{S}$: ?] For parameters $\lambda_1,\lambda_2,\xi_1,\xi_2$ and $r,s\geq 0$,

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}}\left((-1)^{i_2+j_2}\lambda_1^{i_1}\lambda_2^{i_2}\xi_1^{j_1}\xi_2^{j_2}+(\lambda_1^{i_1}\xi_1^{j_1}+\lambda_2^{i_1}\xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2}\right)\times\sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}}\zeta_{\mathcal{F}}(\mathbf{k},\mathbf{l})=0.$$

Kamano's idea: Expansions in two ways

$$K_{r,s}(X):=\frac{1}{r!s!}\int_{\substack{0<t<X\\0<u<X}}\left(\lambda_1\int_t^X\frac{dt'}{1-t'}+\lambda_2\int_u^X\frac{du'}{1-u'}\right)^r\left(\xi_1\int_t^X\frac{dt'}{t'}+\xi_2\int_u^X\frac{du'}{u'}\right)^s\frac{dt\,du}{(1-t)(1-u)}.$$

Binomial expansion $\longrightarrow$ $I_{\mathrm{dch}}(0;\mathrm{bs}(\mathbf{k});X)\times I_{\mathrm{dch}}(0;\mathrm{bs}(\mathbf{l});X)$ $\xrightarrow[p-m\equiv -m \pmod p]{\mathrm{Coeff}_p(K_{r,s}(X)\frac{X}{1-X})}$ $(-1)^{\mathrm{wt}(\mathbf{l})}\zeta_p(\mathbf{k},\bar{\mathbf{l}})$ $\downarrow$

$K_{r,s}(X)$: dch $\qquad\qquad$ WSFs for $\zeta_{\mathcal{A}}$

$\displaystyle\int_{0<t<u<X}+\int_{0<u<t<X}$ $\longrightarrow$ $I_{\mathrm{dch}}(0;\mathrm{bs}(\mathbf{k}),\mathrm{bs}(\mathbf{l});X)$ $\xrightarrow{\mathrm{Coeff}_p(K_{r,s}(X)\frac{X}{1-X})}$ $\zeta_p(\mathbf{k},\mathbf{l})$ $\uparrow$

Question $\qquad \zeta_{\mathcal{A}}$: dch $\overset{?}{\longleftrightarrow}$ $\zeta_{\mathcal{S}}:\beta\Longrightarrow$

Weighted sum formulas for $\zeta_{\mathcal{F}}$ $\quad (\mathcal{F} \in \{\mathcal{A}, \mathcal{S}\})$

WSFs for $\zeta_{\mathcal{F}}$ [$\mathcal{A}$: Kamano(2018), $\mathcal{S}$: ?] For parameters $\lambda_1, \lambda_2, \xi_1, \xi_2$ and $r, s \in \mathbb{Z}_{\geq 0}$

$$\sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} \left((-1)^{i_2+j_2} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} + (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1})(\lambda_1 + \lambda_2)^{i_2} (\xi_1 + \xi_2)^{j_2} \right) \times \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{\mathcal{F}}(\mathbf{k}, \mathbf{l}) = 0.$$

Kamano's idea: Expansions in two ways

$$K_{r,s}(X) := \frac{1}{r!s!} \int_{\substack{0<t<X \\ 0<u<X}} \left(\lambda_1 \int_t^X \frac{dt'}{1-t'} + \lambda_2 \int_u^X \frac{du'}{1-u'} \right)^r \left(\xi_1 \int_t^X \frac{dt'}{t'} + \xi_2 \int_u^X \frac{du'}{u'} \right)^s \frac{dt\,du}{(1-t)(1-u)}.$$

Binomial expansion $\longrightarrow$ $I_{\mathrm{dch}}(0; \mathrm{bs}(\mathbf{k}); X) \times I_{\mathrm{dch}}(0; \mathrm{bs}(\mathbf{l}); X)$ $\xrightarrow[p-m \equiv -m \pmod p]{\mathrm{Coeff}_p(K_{r,s}(X)\frac{X}{1-X})}$ $(-1)^{\mathrm{wt}(\mathbf{l})} \zeta_p(\mathbf{k}, \overline{\mathbf{l}})$

$|$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

$K_{r,s}(X)$: dch $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ WSFs for $\zeta_{\mathcal{A}}$

$|$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\uparrow$

$\displaystyle\int_{0<t<u<X} + \int_{0<u<t<X}$ $\longrightarrow$ $I_{\mathrm{dch}}(0; \mathrm{bs}(\mathbf{k}), \mathrm{bs}(\mathbf{l}); X)$ $\xrightarrow{\mathrm{Coeff}_p(K_{r,s}(X)\frac{X}{1-X})}$ $\zeta_p(\mathbf{k}, \mathbf{l})$

Question $\qquad \zeta_{\mathcal{A}}$: dch $\overset{?}{\longleftrightarrow}$ $\zeta_{\mathcal{S}}$: β $\Longrightarrow$ YES!!

Contents

Weighted sum formulas for $\zeta_{\mathcal{S}}$

Main Theorem [WSFs for $\zeta_{\mathcal{S}}$] For parameters $\lambda_1, \lambda_2, \xi_1, \xi_2$ and $r, s \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} \left((-1)^{i_2+j_2} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} + (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2} \right) \times \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{\mathcal{S}}^{ш}(\mathbf{k}, \mathbf{l}) = 0.$$

Remark. These formulas hold without modulo $\zeta(2)\mathcal{Z}$.

Idea: Expansions in two ways

$$I_{r,s}(X) := \frac{1}{r!s!} \int_{\substack{\epsilon<t<X \\ \epsilon<u<X}} \left(\lambda_1 \int_t^X \frac{d\beta_n(t')}{\beta_n(t')-1} + \lambda_2 \int_u^X \frac{d\beta_n(u')}{\beta_n(u')-1} \right)^r \times \left(\xi_1 \int_t^X \frac{d\beta_n(t')}{\beta_n(t')} + \xi_2 \int_u^X \frac{d\beta_n(u')}{\beta_n(u')} \right)^s \frac{d\beta_n(t) d\beta_n(u)}{(\beta_n(t)-1)(\beta_n(u)-1)}.$$

Comparison the proof of WSFs for $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{S}}$

Binomial expansion $\longrightarrow$ $I_{\mathrm{dch}}(0;\mathrm{bs}(\mathbf{k});X) \times I_{\mathrm{dch}}(0;\mathrm{bs}(\mathbf{l});X)$ $\xrightarrow[\mathrm{Coeff}_p(K_{r,s}(X)\frac{X}{1-X})]{m\mapsto p-m,\ p-m\equiv -m \pmod p}$ $(-1)^{\mathrm{wt}(\mathbf{l})}\zeta_p(\mathbf{k},\bar{\mathbf{l}})$ $\downarrow$ WSFs for $\zeta_{\mathcal{A}}$

$K_{r,s}(X)$: dch

$$\int_{0<t<u<X} + \int_{0<u<t<X} \longrightarrow I_{\mathrm{dch}}(0;\mathrm{bs}(\mathbf{k}),\mathrm{bs}(\mathbf{l});X) \xrightarrow[\mathrm{Coeff}_p(K_{r,s}(X)\frac{X}{1-X})]{} \zeta_p(\mathbf{k},\mathbf{l})$$

$\zeta_p(\mathbf{k},\mathbf{l})$ $\uparrow$ WSFs for $\zeta_{\mathcal{A}}$

Binomial expansion $\longrightarrow$ $I_{\beta_n}(\beta_n(\epsilon);\mathrm{bs}(\mathbf{k});\beta_n(X)) \times I_{\beta_n}(\beta_n(\epsilon);\mathrm{bs}(\mathbf{l});\beta_n(X))$ $\xrightarrow[\int_{0<X<1} I_{r,s}(X)\frac{d\beta_n(X)}{\beta_n(X)-1}]{\beta_n^{-1}(t)=\beta_n(1-t),\ \text{Transformation of integral}}$ $(-1)^{\mathrm{wt}(\mathbf{l})}\zeta_{RS}(\mathbf{k},\bar{\mathbf{l}};T) + P_{r,s}(T)$ $\downarrow$ WSFs for $\zeta_{\mathcal{S}}$

$I_{r,s}(X)$: β_n

$$\int_{\epsilon<t<u<X} + \int_{\epsilon<u<t<X} \to I_{\beta_n}(\beta_n(\epsilon);\mathrm{bs}(\mathbf{k}),\mathrm{bs}(\mathbf{l});\beta_n(X)) \xrightarrow[\int_{0<X<1} I_{r,s}(X)\frac{d\beta_n(X)}{\beta_n(X)-1}]{} \zeta_{RS}(\mathbf{k},\mathbf{l};T)$$

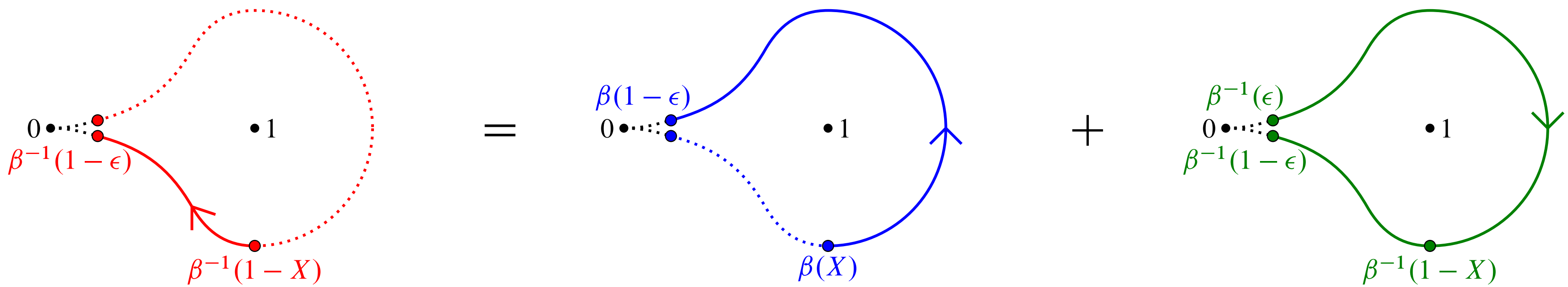
$\zeta_{RS}(\mathbf{k},\mathbf{l};T)$ $\uparrow$ WSFs for $\zeta_{\mathcal{S}}$

Proof (1/2) ($n = 1$ for simplicity)

$$I_\beta(\beta(\epsilon);\mathrm{bs}(\mathbf{k});\beta(X)) \times I_\beta(\beta(\epsilon);\mathrm{bs}(\mathbf{l});\beta(X)) \xrightarrow[\int_{\epsilon<X<1-\epsilon} I_{r,s}(X)\frac{d\beta(X)}{\beta(X)-1}]{\beta^{-1}(t)=\beta(1-t)\ \text{Transformation of integral}} (-1)^{\mathrm{wt}(\mathbf{l})}\zeta_{RS}(\mathbf{k},\bar{\mathbf{l}};T) + P_{r,s}(T).$$

$$I_\beta(\beta(\epsilon);\mathrm{bs}(\mathbf{l});\beta(X)) \to (-1)^{\mathrm{wt}(\mathbf{l})} I_{\beta^{-1}}(\beta^{-1}(1-X);\overline{\mathrm{bs}(\mathbf{l})};\beta^{-1}(1-\epsilon))$$

$$= (-1)^{\mathrm{wt}(\mathbf{l})}\sum_{i=0}^{i_2+j_2+1} I_\beta(\beta(X);\overline{\mathrm{bs}(\mathbf{l})^{i+1}};\beta(1-\epsilon))\, I_{\beta^{-1}}(\beta^{-1}(\epsilon);\overline{\mathrm{bs}(\mathbf{l})_i};\beta^{-1}(1-\epsilon))$$



$$-m = p - m + -p$$

$\beta \rightsquigarrow \beta_n$,
$2\pi i n \rightsquigarrow T$.

$$(-1)^{\mathrm{wt}(\mathbf{l})}\zeta_{RS}(\mathbf{k},\bar{\mathbf{l}};T) + \sum_{h=0}^{i_2}\sum_{m=0}^{l_h-1}\frac{(-1)^{k+h+1}}{T}L(e_{k_0}\cdots e_{k_{i_1}}e_{l_{i_2}}\cdots e_{l_h-m};T)L(e_{l_0-1}\cdots e_{l_{h-1}}e_{m+1};T).$$

$\bar{\mathbf{a}} := (a_m,\dots,a_1)$, $\mathbf{a}_i := (a_1,\dots,a_i)$, $\mathbf{a}^i := (a_i,\dots,a_m)$ for $\mathbf{a} = (a_1,\dots,a_m) \in \{0,1\}$.

Proof (2/2)

These calculation in terms of $I_{r,s}(X)$:

$$\sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} (-1)^{i_2+j_2+1}\lambda_1^{i_1}\lambda_2^{i_2}\xi_1^{j_1}\xi_2^{j_2} \sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}} \Bigg(\zeta_{RS}(\mathbf{k},\mathbf{l};T)$$

$$+ \sum_{h=0}^{i_2}\sum_{m=0}^{l_h-1} \frac{(-1)^{k+h+1}}{T} L(e_{k_0}\cdots e_{k_{i_1}} e_{l_{i_2}}\cdots e_{l_h-m};T) L(e_{l_0-1}\cdots e_{l_{h-1}} e_{m+1};T)\Bigg)$$

$$= \sum_{\substack{i_1+i_2=r\\ j_1+j_2=s}} (\lambda_1^{i_1}\xi_1^{j_1} + \lambda_2^{i_1}\xi_2^{j_1})(\lambda_1+\lambda_2)^{i_2}(\xi_1+\xi_2)^{j_2} \sum_{\substack{\mathbf{k}\in I(i_1+j_1+1,i_1+1)\\ \mathbf{l}\in I(i_2+j_2+1,i_2+1)}} \zeta_{RS}(\mathbf{k},\mathbf{l};T).$$

Since $\begin{cases} \zeta_{RS}(\mathbf{k};0) = \zeta_{\mathcal{S}}^{ш}(\mathbf{k}), \\ L(w;T) \in T\mathcal{Z}[T], \end{cases}$ green terms vanish and we finally have WSFs for $\zeta_{\mathcal{S}}^{ш}(\mathbf{k})$.

□

Weighted sum formulas for $\zeta_{\mathcal{F}}$ For parameters $\lambda_1, \lambda_2, \xi_1, \xi_2$ and $r, s \in \mathbb{Z}_{\geq 0}$,

$$\sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} \left((-1)^{i_2+j_2} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} + (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1})(\lambda_1 + \lambda_2)^{i_2} (\xi_1 + \xi_2)^{j_2} \right) \times \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{\mathcal{F}}(\mathbf{k}, \mathbf{l}) = 0.$$

Remark. WSFs for $\mathcal{F} = \mathcal{S}$ hold without mod $\zeta(2)\mathcal{Z}$.

Thank you for your kind attention!!