2021/3/9

Several sum formulas related with a congruence of symmetric multiple zeta values

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Several multiple zeta(-star) values (MZ(S)Vs)

MZ(S)V

$$\zeta^{\star}(k_1,\dots,k_r) := \sum_{0<m_1\leq\cdots\leq m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathcal{Z} \quad (k_r>1),$$

$$\zeta_M^{\star}(k_1,\dots,k_r) := \sum_{0<m_1\leq\cdots\leq m_r\leq M} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathbb{Q} \quad (M\in\mathbb{Z}_{>0}).$$

Finite MZV (FMZV)

$$\mathcal{A} := \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right)/\left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right)$$

$$\zeta_{\mathcal{A}}^{\star}(k_1,\dots,k_r) := \left(\zeta_{p-1}^{\star}(k_1,\dots,k_r) \bmod p\right)_p \in \mathcal{A}$$

Symmetric MZV (SMZV)

$$\zeta_{\mathcal{S},M}^{\star}(\mathbf{k}) := \sum_{i=0}^{r}(-1)^{k_r+\cdots+k_{i+1}}\zeta_M^{\star}(k_1,\dots,k_i) \times \zeta_M^{\star}(k_r,\dots,k_{i+1})$$

$$\Rightarrow \zeta_{\mathcal{S}}^{\star}(\mathbf{k}) := \lim_{M\to\infty}\zeta_{\mathcal{S},M}^{\star}(\mathbf{k}) \bmod \zeta(2)\mathcal{Z} \in \mathcal{Z}/\zeta(2)\mathcal{Z}.$$

$$\mathcal{Z}_{\mathcal{A}} := \langle \text{all } \zeta_{\mathcal{A}}(\mathbf{k})\rangle_{\mathbb{Q}} \subset \mathcal{A},$$

$$\mathcal{Z}_{\mathcal{S}} := \langle \text{all } \zeta_{\mathcal{S}}(\mathbf{k})\rangle_{\mathbb{Q}} = \mathcal{Z}/\zeta(2)\mathcal{Z}.$$

Conjecture (Kaneko–Zagier).

$$\mathcal{Z}_{\mathcal{A}} \overset{?}{\simeq} \mathcal{Z}/\zeta(2)\mathcal{Z} : \zeta_{\mathcal{A}}(\mathbf{k}) \leftrightarrow \zeta_{\mathcal{S}}(\mathbf{k}).$$

Several relations for MZSVs ,SMZSVs and FMZSVs

$\bullet \in \{\emptyset, \mathcal{F}\}$ with $\mathcal{F} \in \{\mathcal{A}, \mathcal{S}\}$.

$$I_0(k,r,s) = \left\{ \mathbf{k} : \text{adm.} \;\middle|\; \begin{array}{c} \mathrm{wt}(\mathbf{k}) = k \\ \mathrm{dep}(\mathbf{k}) = r \\ \mathrm{ht}(\mathbf{k}) = s \end{array} \right\},$$

$$\mathfrak{Z}_\bullet(k) = \begin{cases} \zeta(k) & (\bullet = \emptyset), \\ \left(\frac{B_{p-k}}{k} \bmod p\right)_p & (\bullet = \mathcal{A}), \\ \zeta(k) \bmod \zeta(2)\mathcal{Z} & (\bullet = \mathcal{S}), \end{cases}$$

Aoki–Ohno's relation[$\emptyset$: Aoki–Ohno (2005), $\mathcal{A}$: Kaneko–Oyama–Saito(2018), $\mathcal{S}$: ?]

$$\sum_{\mathbf{k} \in I_0(k,*,s)} \zeta_\bullet^\star(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\mathfrak{Z}_\bullet(k) \quad (k, s \in \mathbb{Z}_{>0}).$$

Sum formula[$\emptyset$: Granville (1996), Zagier (1995) $\mathcal{A}$: Saito–Wakabayashi(2015), $\mathcal{S}$: Murahara(2016)]

$$\sum_{\mathbf{k} \in I_0(k,r,*)} \zeta_\bullet^\star(\mathbf{k}) = \left(\binom{k-1}{r-1} + \delta_{\bullet,\mathcal{F}}(-1)^r\right)\mathfrak{Z}_\bullet(k) \quad (k, r \in \mathbb{Z}_{>0}).$$

Generalized height-one duality[$\emptyset$: Li (2012), $\mathcal{A}, \mathcal{S}$: Sakurada(2019)]

For $Q \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \zeta(7), \ldots]$,

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1,n+1,s)} \zeta_\bullet^\star(\mathbf{k}) - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1,m+1,s)} \zeta_\bullet^\star(\mathbf{k}) = \delta_{\bullet,\emptyset} Q \quad (m, n, s \in \mathbb{Z}_{>0}).$$

	$\zeta^\star$	$\zeta^\star_{\mathcal{A}}$	$\zeta^\star_{\mathcal{S}}$
Aoki–Ohno's relation	◯	◯	
Sum formula	◯	◯	◯
Generalized height-one duality	◯	◯	◯

	$\zeta^\star$	$\zeta^\star_{\mathcal{A}}$	$\zeta^\star_{\mathcal{S}}$
Aoki–Ohno's relation	○	○	○
Sum formula	○	○	○
Generalized height-one duality	○	○	○

We prove these relations for $\zeta^\star_{\mathcal{S}}(\mathbf{k})$ at once.

Main Theorem $\displaystyle\sum_{\mathbf{k}\in I_0(k,r,s)} \zeta^\star_{\mathcal{S}}(\mathbf{k}) = \sum_{\mathbf{k}\in I_0(k,r,s)} \zeta^\star(\mathbf{k}) + P_{k,r,s} \bmod \zeta(2)\mathcal{Z}, \quad (k,r,s\in\mathbb{Z}_{>0})$

where $P_{k,r,s}\in\mathbb{Q}[\zeta(3),\zeta(5),\zeta(7),\ldots]$ with $\alpha+\beta=x+y, \alpha\beta=xy-z^2$ is given by

$$\frac{1}{\alpha\beta}\left(\frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\beta)\Gamma(1-y+\beta)}-1\right)=\sum_{k,r,s\geq 0} P_{k,r,s}x^{k-r-s}y^{r-s}z^{2s-2}$$

$$\in \mathcal{Z}[[x,y,z]] \bmod \zeta(2)\mathcal{Z}.$$

Remark. $\displaystyle\Gamma(1+x)=\exp\left(-\gamma x+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}x^n\right) \quad (|x|<1).$

- **Aoki–Ohno's relation** $\cdots$ If $x=y \Rightarrow P_{k,r,s}=0 \bmod \zeta(2)\mathcal{Z}$.

$$\sum_{\mathbf{k}\in I_0(k,*,s)} \zeta^\star_{\mathcal{S}}(\mathbf{k}) = \sum_{\mathbf{k}\in I_0(k,*,s)} \zeta^\star(\mathbf{k}) + 0 = 2\binom{k-1}{2s-1}(1-2^{1-k})\zeta(k).$$

- **Sum formula** $\cdots$ If $z^2=xy \Rightarrow P_{k,r,s}=(-1)^r\zeta(k) \bmod \zeta(2)\mathcal{Z}$.

$$\sum_{\mathbf{k}\in I_0(k,r,*)} \zeta^\star_{\mathcal{S}}(\mathbf{k}) = \sum_{\mathbf{k}\in I_0(k,r,*)} \zeta^\star(\mathbf{k}) + (-1)^r\zeta(k) = \left(\binom{k-1}{r-1}+(-1)^r\right)\zeta(k).$$

- **Generalized height-one duality** $\cdots$ $\mathcal{S}$ analogue of the generalized height-one duality vanishes because of $P_{k,r,s}$.

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Sketch of the proof of the Main Theorem

Definition. For $\mathbf{k} = (k_1, \ldots, k_r)$, $M \in \mathbb{Z}_{>0}$, we define

$$\mathscr{L}^{\star}_{M}(\mathbf{k};t) := \sum_{\substack{m_1 \preceq \cdots \preceq m_r \\ 0<|m_1|,\ldots,|m_{r-1}|\leq M \\ -M \leq m_r < 0}} \frac{t^{-m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}[t],$$

where "$\prec$": $1 \prec 2 \prec \cdots \prec (\infty = -\infty) \prec \cdots \prec -2 \prec -1$.

Remark. $\mathscr{L}^{\star}_{M}(\mathbf{k};0) = 0$, $\quad \mathscr{L}^{\star}_{M}(\mathbf{k};1) = \zeta^{\star}_{\mathcal{S},M}(\mathbf{k}) - \zeta^{\star}_{M}(\mathbf{k})$.

And we consider a differential equation of the following generating function:

$$\begin{cases} \Delta^{\star}_{0,M}(t) = \displaystyle\sum_{k,r,s\geq 0} \left(\sum_{\mathbf{k}\in I_0(k,r,s)} \mathscr{L}^{\star}_{M}(\mathbf{k};t) \right) x^{k-r-s} y^{r-s} z^{2s-2}, \\ \Delta^{\star}_{0,M}(0) = 0. \end{cases}$$

Differential equation of $\varDelta^{\star}_{0,M}(t)$

$$t(1-t)\frac{d^2\varDelta^{\star}_{0,M}(t)}{dt^2}+\Big((1-t)(1+x)-ty\Big)\frac{d\varDelta^{\star}_{0,M}(t)}{dt}-(xy-z^2)\varDelta^{\star}_{0,M}(t)$$
$$=\varPhi^{\star}_{\mathcal{S},M}-t^M\varPhi^{\star}_M,$$

Unique solution: $\quad \varDelta^{\star}_{0,M}(t)=\varPhi^{\star}_{\mathcal{S},M}u_1(t)-\varPhi^{\star}_M u_{2,M}(t).$

$$\varPhi^{\star}_{\bullet,M}=1+\sum_{\substack{k,r,s\geq 0\\(k,r,s)\neq(0,0,0)}}\Bigg(\sum_{\mathbf{k}\in I(k,r,s)}\zeta^{\star}_{\bullet,M}(\mathbf{k})\Bigg)x^{k-r-s}y^{r-s}z^{2s}\quad(\bullet\in\{\emptyset,\mathcal{S}\}),$$

$$u_1(t)=\frac{1}{\alpha\beta}\left({}_2F_1\left(\begin{matrix}\alpha,\beta\\x+1\end{matrix};t\right)-1\right),$$

$$u_{2,M}(t)=\frac{t^M}{(\alpha+M)(\beta+M)}\left({}_3F_2\left(\begin{matrix}\alpha+M,\beta+M,1\\x+M+1,M+1\end{matrix};t\right)-1\right),$$

where $\begin{cases}\alpha+\beta=x+y,\\ \alpha\beta=xy-z^2,\end{cases}$

Put $t = 1$ and take the limit $M \to \infty$,

$$\Delta^{\star}_{0,M}(1) = \underbrace{\Phi^{\star}_{\mathcal{S},M} u_1(1)}_{\to \Phi^{\star}_{\mathcal{S}} u_1(1) \equiv u_1(1)} - \underbrace{\Phi^{\star}_{M} u_{2,M}(1)}_{\to 0}$$

Put $t = 1$ and take the limit $M \to \infty$,

$$\Delta^{\star}_{\mathbf{0},M}(1) = \underbrace{\Phi^{\star}_{\mathcal{S},M} u_1(1)}_{\to \Phi^{\star}_{\mathcal{S}} u_1(1) \equiv u_1(1)} - \underbrace{\Phi^{\star}_{M} u_{2,M}(1)}_{\to 0}.$$

$$\begin{cases} \mathrm{Coeff}(\Phi^{\star}_{M}; x^{k-r-s} y^{r-s} z^{2s}) = \displaystyle\sum_{\mathbf{k} \in I(k,r,s)} \zeta^{\star}_{M}(k_1, \dots, k_r) \\ \qquad \le C\left(1 + \displaystyle\int_1^M \frac{1}{m} dm\right)^r = O(\log^r M) \quad (\exists C > 0), \\ \mathrm{Coeff}(u_{2,M}(1); x^{k-r-s} y^{r-s} z^{2s-2}) = O\left(\dfrac{1}{M}\right), \end{cases}$$

$$\Longrightarrow \mathrm{Coeff}(\Phi^{\star}_{M} u_{2,M}(1); x^{k-r-s} y^{r-s} z^{2s}) = O\left(\frac{\log^r M}{M}\right) \to 0 \ (M \to \infty).$$

Put $t = 1$ and take the limit $M \to \infty$,

$$\Delta^{\star}_{0,M}(1) = \underbrace{\Phi^{\star}_{\mathcal{S},M} u_1(1)}_{\to \Phi^{\star}_{\mathcal{S}} u_1(1) \equiv u_1(1)} - \underbrace{\Phi^{\star}_{M} u_{2,M}(1)}_{\to 0}.$$

$$\lim_{M\to\infty} \Delta^{\star}_{0,M}(1) = \sum_{k,r,s\geq 0} \Bigg(\sum_{\mathbf{k}\in I_0(k,r,s)} \Big(\zeta^{\star}_{\mathcal{S}}(\mathbf{k}) - \zeta^{\star}(\mathbf{k}) \Big) \Bigg) x^{k-r-s} y^{r-s} z^{2s-2},$$

$$\lim_{M\to\infty} \Phi^{\star}_{\mathcal{S},M} u_1(1) = \Bigg(1 + \sum_{\substack{k,r,s\geq 0 \\ (k,r,s)\neq 0}} \underbrace{\Bigg(\sum_{\mathbf{k}\in I(k,r,s)} \zeta^{\star}_{\mathcal{S}}(\mathbf{k}) \Bigg)}_{\equiv 0} x^{k-r-s} y^{r-s} z^{2s} \Bigg) u_1(1)$$

$$= \frac{1}{\alpha\beta} \left(\frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\beta)\Gamma(1-y+\beta)} - 1 \right) \bmod \zeta(2)\mathcal{Z}$$

$$= \sum_{k,r,s\geq 0} P_{k,r,s} x^{k-r-s} y^{r-s} z^{2s-2} \bmod \zeta(2)\mathcal{Z}.$$

$$(\alpha + \beta = x + y, \alpha\beta = xy - z^2)$$

Comparing these equations implies Main Theorem. □

Summary

Main Theorem.

$$\sum_{\mathbf{k}\in I_0(k,r,s)} \zeta^{\star}_{\mathcal{S}}(\mathbf{k}) = \sum_{\mathbf{k}\in I_0(k,r,s)} \zeta^{\star}(\mathbf{k}) + P_{k,r,s} \bmod \zeta(2)\mathcal{Z}.$$

Set $\bullet \in \{\emptyset, \mathcal{S}\}$, $Q \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \zeta(7), \ldots]$,

$$\text{Aoki–Ohno's relation}: \sum_{\mathbf{k}\in I_0(k,*,s)} \zeta^{\star}_{\bullet}(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\zeta(k),$$

$$\text{Sum formula}: \sum_{\mathbf{k}\in I_0(k,r,*)} \zeta^{\star}_{\bullet}(\mathbf{k}) = \left(\binom{k-1}{r-1} + \delta_{\bullet,\mathcal{S}}(-1)^r\right)\zeta(k),$$

$$\text{Generalized heigh-one duality}: (-1)^m \sum_{\mathbf{k}\in I_0(m+n+1,n+1,s)} \zeta^{\star}_{\bullet}(\mathbf{k}) - (-1)^n \sum_{\mathbf{k}\in I_0(m+n+1,m+1,s)} \zeta^{\star}_{\bullet}(\mathbf{k}) = \delta_{\bullet,\emptyset} Q.$$

Remark.

$$\sum_{\mathbf{k}\in I_0(k,r,s)} \zeta^{\star}_{\mathcal{S}}(\mathbf{k}) = (-1)^{k-1} \sum_{\mathbf{k}\in I_0(k,r,s)} (-1)^r \zeta_{\mathcal{S}}(\mathbf{k}).$$