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# Weighted sum formulas and refined symmetric multiple zeta values

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§1 Introduction

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## MZVs and Multi-polylogarithm (MPL)

$$\zeta(k_1, \dots, k_r) := \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \in \mathcal{Z} \quad (k_r > 1),$$

$$\zeta^{\text{III}}(k_1, \dots, k_r) := Z^{\text{III}}(e_{k_1} \dots e_{k_r}; T)|_{T=0} \quad (e_{k_i} := e_1 e_0^{k_i-1} \in \mathbb{Q}\langle e_0, e_1 \rangle),$$

$$\text{Li}_{k_1, \dots, k_r}(X) := \sum_{0 < m_1 < \dots < m_r} \frac{X^{m_r}}{m_1^{k_1} \dots m_r^{k_r}} \quad (0 < X < 1).$$

## Finite MZV (FMZV)

$$\mathcal{A} := (\prod_p \mathbb{Z}/p\mathbb{Z}) / (\bigoplus_p \mathbb{Z}/p\mathbb{Z})$$

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r)$$

$$:= (\zeta_p(k_1, \dots, k_r) \bmod p)_p \in \mathcal{A}$$

$$\mathcal{Z}_{\mathcal{A}} := \langle \text{all } \zeta_{\mathcal{A}}(\mathbf{k}) \rangle_{\mathbb{Q}} \subset \mathcal{A},$$

$$\mathcal{Z}_{\mathcal{S}} := \langle \text{all } \zeta_{\mathcal{S}}(\mathbf{k}) \rangle_{\mathbb{Q}} = \mathcal{Z} / \zeta(2)\mathcal{Z}.$$

## Symmetric MZV (SMZV)

$$\zeta_{\mathcal{S}}^{\text{III}}(\mathbf{k}) := \sum_{i=0}^r (-1)^{k_r + \dots + k_{i+1}} \zeta^{\text{III}}(k_1, \dots, k_i) \\ \times \zeta^{\text{III}}(k_r, \dots, k_{i+1})$$

$$\implies \zeta_{\mathcal{S}}(\mathbf{k}) := \zeta_{\mathcal{S}}^{\text{III}}(\mathbf{k}) \bmod \zeta(2)\mathcal{Z} \in \mathcal{Z} / \zeta(2)\mathcal{Z}.$$

## Conjecture (Kaneko–Zagier).

$$\mathcal{Z}_{\mathcal{A}} \stackrel{?}{\simeq} \mathcal{Z} / \zeta(2)\mathcal{Z} : \zeta_{\mathcal{A}}(\mathbf{k}) \leftrightarrow \zeta_{\mathcal{S}}(\mathbf{k}).$$

In this talk, we use **Tangential base points**

$$0' := 1_0 \quad \begin{array}{c} 0 \\ \bullet \rightarrow \end{array} \quad 1' := (-1)_1 \quad \begin{array}{c} 1 \\ \leftarrow \bullet \end{array}$$

**Definition** (Regularized iterated integrals). Fix  $a_1, \dots, a_k \in \{0, 1\}$  and tangential base points  $x, y \in \{0', 1'\}$ . For  $\gamma : [0, 1] \rightarrow \mathbb{C}$ : path from  $x$  to  $y$  such that  $\gamma((0, 1)) \subset \mathbb{C} \setminus \{0, 1\}$ ,

$$\int_{\epsilon < t_1 < \dots < t_k < 1 - \epsilon} \prod_{i=1}^k \frac{d\gamma(t_i)}{\gamma(t_i) - a_i} = c_0 + \sum_{j=1}^k c_j (\log \epsilon)^j + O(\epsilon \log^{k+1} \epsilon).$$

We define  $I_\gamma(x; a_1, \dots, a_k; y) := c_0$ .

Remark. Fix  $a_0, a_{k+1} \in \{0, 1\}$  with  $a_0 \neq a_1$  and  $a_k \neq a_{k+1}$ . For a path  $\gamma$  from  $a_0$  to  $a_{k+1}$ ,

$$I_\gamma(a_0; a_1, \dots, a_k; a_{k+1}) = \int_{0 < t_1 < \dots < t_k < 1} \prod_{i=1}^k \frac{d\gamma(t_i)}{\gamma(t_i) - a_i}.$$

Let dch be a straight path from 0 to 1:

$$\begin{array}{c} \bullet \longrightarrow \bullet \\ 0 \qquad \qquad 1 \end{array} : \text{dch}(t) := t \quad (t \in [0, 1])$$

(dch: **d**roit **ch**emin = straight path).

## MZVs and MPL

$$(-1)^r \zeta(k_1, \dots, k_r) = I_{\text{dch}}(0; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; 1) \in \mathcal{Z} \quad (k_r > 1),$$

$$(-1)^r \zeta^{\text{III}}(k_1, \dots, k_r) = I_{\text{dch}}(0'; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; 1') \in \mathcal{Z},$$

$$\text{Li}_{k_1, \dots, k_r}(X) = I_{\text{dch}}(0; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; X) \quad (0 < X < 1).$$

## FMZV

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r)$$

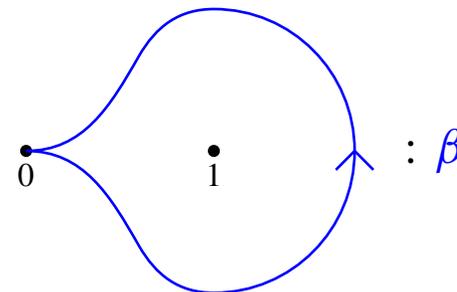
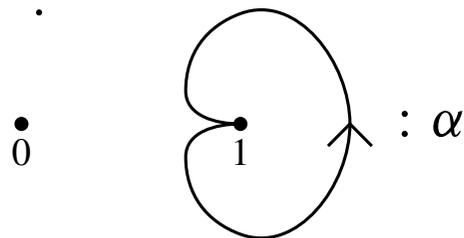
$$= \left( \text{Coeff}_p \left( \frac{I_{\text{dch}}(0; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; X)}{1 - X} \right) \right)_p$$

## SMZV

What is  $\zeta_{\mathcal{S}}(\mathbf{k})$  in terms of iterated integral?

$\Rightarrow$  **Refined SMZV!**

Set a path  $\beta := \text{dch} \cdot \alpha \cdot \text{dch}^{-1}$ .



**Definition** (Hirose, RSMZV). For an index  $\mathbf{k} = (k_1, \dots, k_r)$ , define

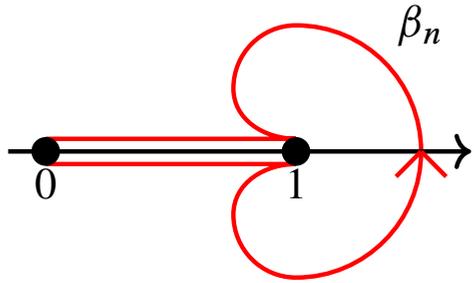
$$\begin{aligned} \zeta_{RS}(\mathbf{k}) &:= \frac{(-1)^r}{2\pi i} I_{\beta}(0'; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}, \mathbf{1}; 0') \\ &= \sum_{\substack{0 \leq a \leq b \leq r \\ k_j = 1 \ (a < \forall j \leq b)}} \frac{(-2\pi i)^{b-a}}{(b-a+1)!} (-1)^{k_r + \dots + k_{b+1}} \zeta^{\text{III}}(k_1, \dots, k_a) \zeta^{\text{III}}(k_r, \dots, k_{b+1}) \\ &\in \mathcal{Z}[2\pi i]. \end{aligned}$$

Remark.  $\zeta_{RS}(\mathbf{k})$  is a lift of  $\zeta_{\mathcal{S}}(\mathbf{k})$ :

$$\zeta_{RS}(\mathbf{k}) \equiv \sum_{i=0}^r (-1)^{k_r + \dots + k_{i+1}} \zeta^{\text{III}}(k_1, \dots, k_i) \zeta^{\text{III}}(k_r, \dots, k_{i+1}) \pmod{2\pi i \mathcal{Z}[2\pi i]}.$$

Hirose: DSR, duality, ... etc. for  $\zeta_{\mathcal{S}}(\mathbf{k})$  by using  $\zeta_{RS}(\mathbf{k})$ .

Set a path  $\beta_n := \text{dch} \cdot \alpha^n \cdot \text{dch}^{-1}$  for  $n \in \mathbb{Z}_{>0}$  and a linear map  $L_n : \mathbb{Q}\langle e_0, e_1 \rangle \rightarrow \mathbb{C}$  by



$$L_n(e_{k_1} \cdots e_{k_r} e_1) := I_{\beta_n}(0'; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}, 1; 0') \in 2\pi i n \mathcal{Z}[2\pi i n],$$

Then there uniquely exists  $L(w; T) \in T \mathcal{Z}[T]$  for  $w \in \mathbb{Q}\langle e_0, e_1 \rangle$  such that

$$L_n(w) = L(w; 2\pi i n).$$

**Definition** (Hirose).

$$\zeta_{RS}(k_1, \dots, k_r; T) := \frac{(-1)^r}{T} L(e_{k_1} \cdots e_{k_r} e_1; T)$$

Remark.

- $\zeta_{RS}(k_1, \dots, k_r; 2\pi i) = \zeta_{RS}(k_1, \dots, k_r),$
- $\zeta_{RS}(k_1, \dots, k_r; 0) = \zeta_{\mathcal{S}}^{\text{III}}(k_1, \dots, k_r).$

$$\begin{aligned}
(-1)^r \zeta(k_1, \dots, k_r) &= I_{\text{dch}}(0; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; 1) \quad (k_r > 1), \\
(-1)^r \zeta^{\text{III}}(k_1, \dots, k_r) &= I_{\text{dch}}(0'; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; 1'), \\
\text{Li}_{k_1, \dots, k_r}(X) &= I_{\text{dch}}(0; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; X) \quad (0 < X < 1), \\
\zeta_{\mathcal{A}}(k_1, \dots, k_r) &= \left( \text{Coeff}_p \left( \frac{I_{\text{dch}}(0; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}; X)}{1 - X} \right) \right)_p, \\
\zeta_{RS}(k_1, \dots, k_r) &:= \frac{(-1)^r}{2\pi i} I_{\beta}(0'; 1, 0^{k_1-1}, \dots, 1, 0^{k_r-1}, \mathbf{1}; 0'), \\
\zeta_{\mathcal{S}}(k_1, \dots, k_r) &= \zeta_{RS}(k_1, \dots, k_r) \bmod 2\pi i \mathcal{Z}[2\pi i], \\
\zeta_{RS}(k_1, \dots, k_r; T) &:= \frac{(-1)^r}{T} L(e_{k_1} \cdots e_{k_r} \mathbf{e}_1; T), \\
\zeta_{\mathcal{S}}^{\text{III}}(k_1, \dots, k_r) &= \zeta_{RS}(k_1, \dots, k_r; 0).
\end{aligned}$$

**WSFs for  $\zeta_{\mathcal{F}}$**  [ $\mathcal{A}$ : Kamano (2018),  $\mathcal{S}$ : ?] For parameters  $\lambda_1, \lambda_2, \xi_1, \xi_2$  and  $r, s \in \mathbb{Z}_{\geq 0}$ ,

$$\sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} \left( (-1)^{i_2+j_2} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} + (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1}) (\lambda_1 + \lambda_2)^{i_2} (\xi_1 + \xi_2)^{j_2} \right) \times \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{\mathcal{F}}(\mathbf{k}, \mathbf{l}) = 0.$$

For  $\mathbf{k} = (k_1, \dots, k_r)$ , set  $w(\mathbf{k}) := \begin{cases} 0 & (k_1 > 1), \\ m & (k_1 = \dots = k_m = 1, k_{m+1} > 1). \end{cases}_{k-1}$

$$(\lambda_1, \lambda_2, \xi_1, \xi_2) = (1, 0, 0, 1) \Rightarrow \sum_{\mathbf{k} \in I(k+r, k)} w(\mathbf{k}) \zeta_{\mathcal{F}}(\mathbf{k}) = (-1)^{r-1} \zeta_{\mathcal{F}}(\overbrace{1, \dots, 1}^{k-1}, r+1) \quad (k \in \mathbb{Z}_{>0}, r \in \mathbb{Z}_{\geq 0}),$$

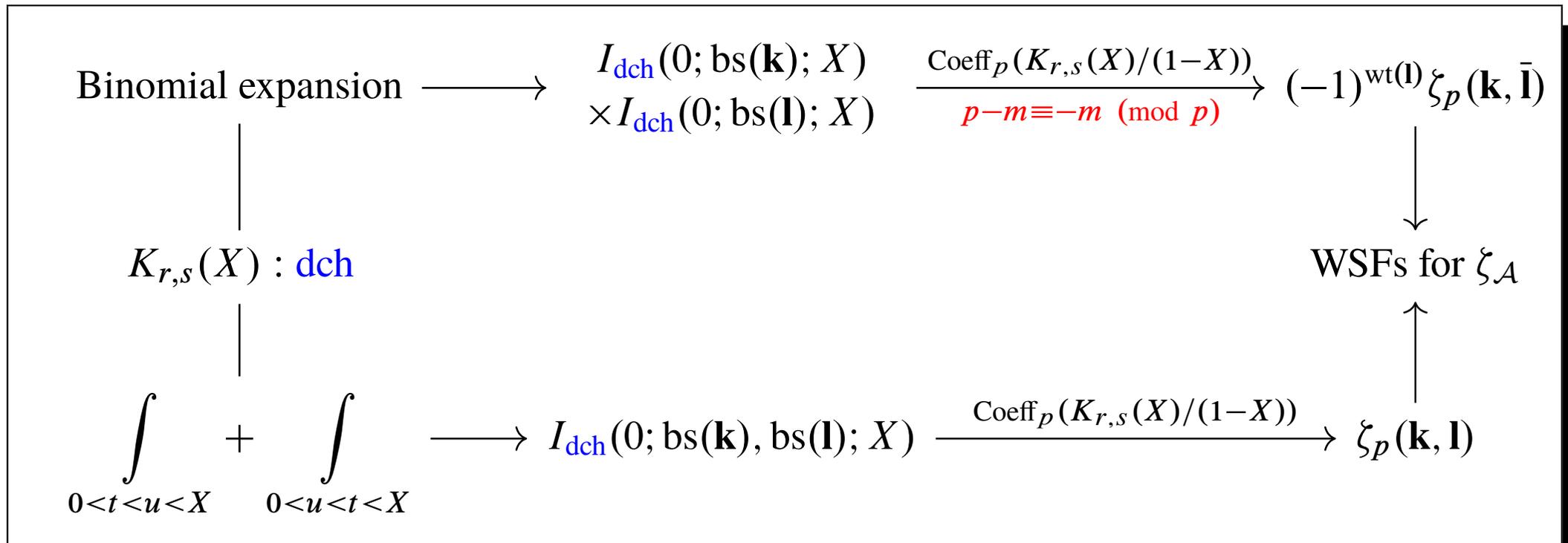
$$(\lambda_1, \lambda_2, \xi_1, \xi_2) = (1, 1, -1, 1) \Rightarrow \sum_{\mathbf{k} \in I(k+r, k)} 2^{w(\mathbf{k})} \zeta_{\mathcal{F}}(\mathbf{k}) = 0 \quad (k \in \mathbb{Z}_{>0}, 2 \leq r : \text{even}).$$

Remark. These WSFs for  $\zeta_{\mathcal{S}}$  are not yet proved.

**Kamano's idea:** Expansions in two ways

$$K_{r,s}(X) := \frac{1}{r!s!} \int_{\substack{0 < t < X \\ 0 < u < X}} \left( \lambda_1 \int_t^X \frac{dt'}{1-t'} + \lambda_2 \int_u^X \frac{du'}{1-u'} \right)^r \left( \xi_1 \int_t^X \frac{dt'}{t'} + \xi_2 \int_u^X \frac{du'}{u'} \right)^s \frac{dt du}{(1-t)(1-u)}.$$

For  $\mathbf{k} = (k_1, \dots, k_r)$ , set  $\text{bs}(\mathbf{k}) := (1, 0^{k_1-1}, \dots, 1, 0^{k_r-1})$ .



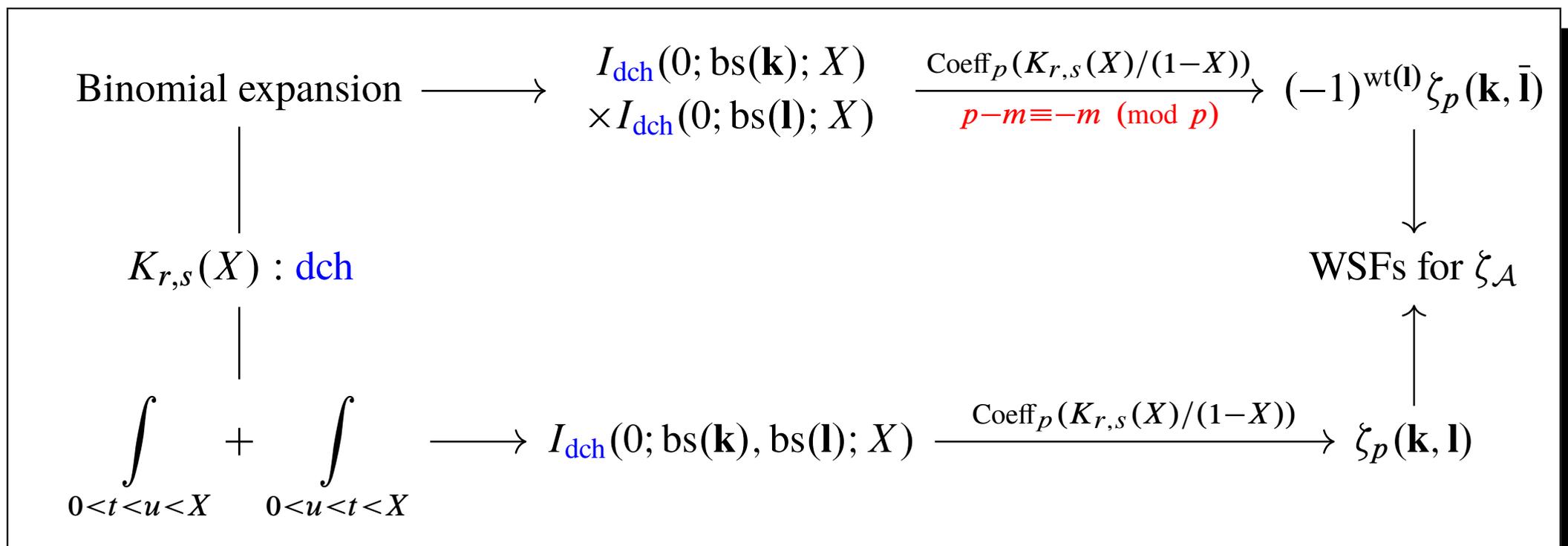
**Question**

$$\zeta_{\mathcal{A}} : \text{dch} \overset{?}{\longleftrightarrow} \zeta_{\mathcal{S}} : \beta \implies$$

**Kamano's idea:** Expansions in two ways

$$K_{r,s}(X) := \frac{1}{r!s!} \int_{\substack{0 < t < X \\ 0 < u < X}} \left( \lambda_1 \int_t^X \frac{dt'}{1-t'} + \lambda_2 \int_u^X \frac{du'}{1-u'} \right)^r \left( \xi_1 \int_t^X \frac{dt'}{t'} + \xi_2 \int_u^X \frac{du'}{u'} \right)^s \frac{dt du}{(1-t)(1-u)}.$$

For  $\mathbf{k} = (k_1, \dots, k_r)$ , set  $\text{bs}(\mathbf{k}) := (1, 0^{k_1-1}, \dots, 1, 0^{k_r-1})$ .



**Question**

$\zeta_{\mathcal{A}} : \text{dch} \overset{?}{\longleftrightarrow} \zeta_{\mathcal{S}} : \beta \implies \text{YES!!}$

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**§2 Main result**

§3 Sketch of the proof

**Main Theorem** [WSFs for  $\zeta_{\mathcal{S}}$ ] For parameters  $\lambda_1, \lambda_2, \xi_1, \xi_2$  and  $r, s \in \mathbb{Z}_{\geq 0}$ ,

$$\sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} \left( (-1)^{i_2+j_2} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} + (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1}) (\lambda_1 + \lambda_2)^{i_2} (\xi_1 + \xi_2)^{j_2} \right) \times \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{\mathcal{S}}^{\text{III}}(\mathbf{k}, \mathbf{l}) = 0.$$

Remark. These formulas hold **without modulo**  $\zeta(2)\mathbb{Z}$ .

For  $\mathbf{k} = (k_1, \dots, k_r)$ , set  $w(\mathbf{k}) := \begin{cases} 0 & (k_1 > 1), \\ m & (k_1 = \dots = k_m = 1, k_{m+1} > 1). \end{cases}$

$$(\lambda_1, \lambda_2, \xi_1, \xi_2) = (1, 0, 0, 1) \Rightarrow \sum_{\mathbf{k} \in I(k+r, k)} w(\mathbf{k}) \zeta_{\mathcal{S}}(\mathbf{k}) = (-1)^{r-1} \zeta_{\mathcal{S}}(\overbrace{1, \dots, 1}^{k-1}, r+1) \quad (k \in \mathbb{Z}_{>0}, r \in \mathbb{Z}_{\geq 0}),$$

$$(\lambda_1, \lambda_2, \xi_1, \xi_2) = (1, 1, -1, 1) \Rightarrow \sum_{\mathbf{k} \in I(k+r, k)} 2^{w(\mathbf{k})} \zeta_{\mathcal{S}}(\mathbf{k}) = 0 \quad (k \in \mathbb{Z}_{>0}, 2 \leq r : \text{even}).$$

# Contents

§1 Introduction

§2 Main result

**§3 Sketch of the proof**

$$\begin{array}{ccc}
\text{Binomial expansion} & \longrightarrow & I_{\text{dch}}(0; \text{bs}(\mathbf{k}); X) \\
& & \times I_{\text{dch}}(0; \text{bs}(\mathbf{l}); X) \xrightarrow[\text{Coeff}_p(K_{r,s}(X)/(1-X))]{m \mapsto p-m, p-m \equiv -m \pmod{p}} (-1)^{\text{wt}(\mathbf{l})} \zeta_p(\mathbf{k}, \bar{\mathbf{l}}) \\
& & \downarrow \\
& & \text{WSFs for } \zeta_{\mathcal{A}} \\
& & \uparrow \\
\int_{0 < t < u < X} + \int_{0 < u < t < X} & \longrightarrow & I_{\text{dch}}(0; \text{bs}(\mathbf{k}), \text{bs}(\mathbf{l}); X) \xrightarrow[\text{Coeff}_p(K_{r,s}(X)/(1-X))]{\phantom{m \mapsto p-m, p-m \equiv -m \pmod{p}}} \zeta_p(\mathbf{k}, \mathbf{l})
\end{array}$$

$K_{r,s}(X) : \text{dch}$

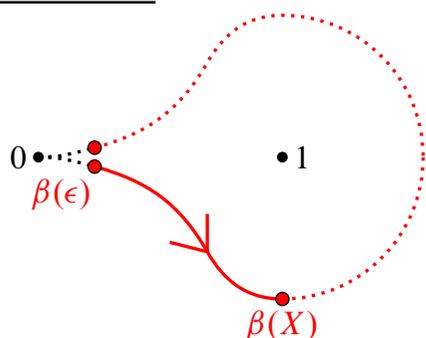
$$I_{r,s}(X) := \frac{1}{r!s!} \int_{\substack{\epsilon < t < X \\ \epsilon < u < X}} \left( \lambda_1 \int_t^X \frac{d\beta_n(t')}{\beta_n(t') - 1} + \lambda_2 \int_u^X \frac{d\beta_n(u')}{\beta_n(u') - 1} \right)^r \times \left( \xi_1 \int_t^X \frac{d\beta_n(t')}{\beta_n(t')} + \xi_2 \int_u^X \frac{d\beta_n(u')}{\beta_n(u')} \right)^s \frac{d\beta_n(t) d\beta_n(u)}{(\beta_n(t) - 1)(\beta_n(u) - 1)}.$$

$$\begin{array}{ccc}
\text{Binomial expansion} & \longrightarrow & I_{\beta_n}(\beta_n(0); \text{bs}(\mathbf{k}); \beta_n(X)) \\
& & \times I_{\beta_n}(\beta_n(0); \text{bs}(\mathbf{l}); \beta_n(X)) \xrightarrow[\int_{0 < X < 1} I_{r,s}(X) \frac{d\beta_n(X)}{\beta_n(X) - 1}]{\beta_n^{-1}(t) = \beta_n(1-t), \text{ Transformation of integral}} (-1)^{\text{wt}(\mathbf{l})} \zeta_{RS}(\mathbf{k}, \bar{\mathbf{l}}; T) \\
& & \downarrow \\
& & \text{WSFs for } \zeta_{\mathcal{S}} \\
& & \uparrow \\
\int_{0 < t < u < X} + \int_{0 < u < t < X} & \Rightarrow & I_{\beta_n}(\beta_n(0); \text{bs}(\mathbf{k}), \text{bs}(\mathbf{l}); \beta_n(X)) \xrightarrow[\int_{0 < X < 1} I_{r,s}(X) \frac{d\beta_n(X)}{\beta_n(X) - 1}]{\phantom{\beta_n^{-1}(t) = \beta_n(1-t), \text{ Transformation of integral}}} \zeta_{RS}(\mathbf{k}, \mathbf{l}; T)
\end{array}$$

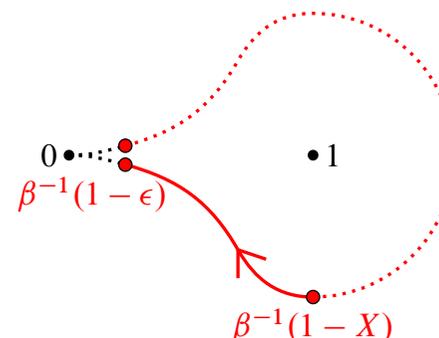
$I_{r,s}(X) : \beta_n$

$$\begin{aligned}
 & I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) \\
 & \times I_\beta(\beta(\epsilon); \text{bs}(\mathbf{l}); \beta(X)) \xrightarrow[\int_{\epsilon < X < 1-\epsilon} I_{r,s}(X) \frac{d\beta(X)}{\beta(X)^{-1}}]{\substack{\beta^{-1}(t) = \beta(1-t) \\ \text{Transformation of integral}}} (-1)^{\text{wt}(\mathbf{l})} \zeta_{RS}(\mathbf{k}, \bar{\mathbf{l}}; T) + P_{r,s}(T)
 \end{aligned}$$

**Point 1:**



$$\beta(t) = \beta^{-1}(1 - t)$$

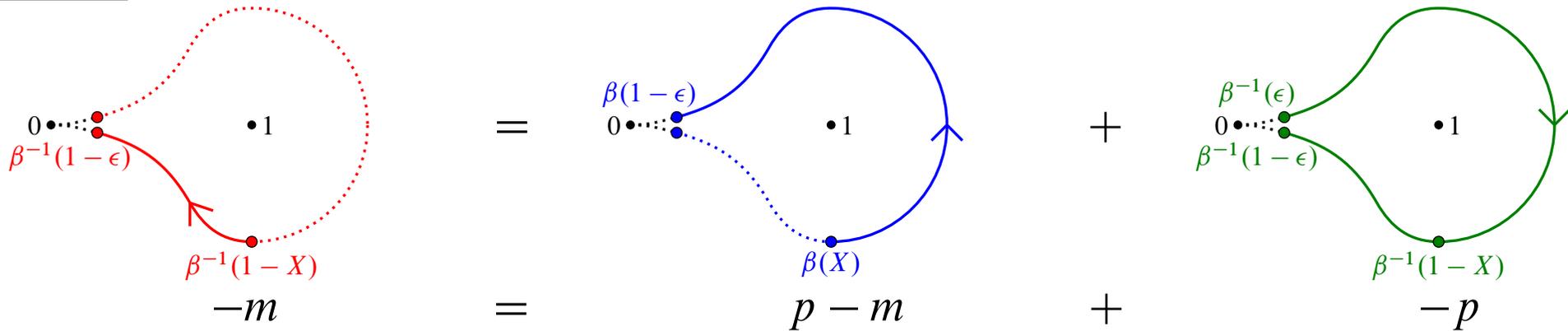


$$\begin{aligned}
 & I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) I_\beta(\beta(\epsilon); \text{bs}(\mathbf{l}); \beta(X)) \\
 & = (-1)^{\text{wt}(\mathbf{l})} I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) I_{\beta^{-1}}(\beta^{-1}(1 - X); \overline{\text{bs}(\mathbf{l})}; \beta^{-1}(1 - \epsilon)),
 \end{aligned}$$

where  $\bar{\mathbf{a}} := (a_m, \dots, a_1)$  for  $\mathbf{a} = (a_1, \dots, a_m) \in \{0, 1\}^m$ .

$$I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) \times I_\beta(\beta(\epsilon); \text{bs}(\mathbf{l}); \beta(X)) \xrightarrow[\int_{\epsilon < X < 1-\epsilon} I_{r,s}(X) \frac{d\beta(X)}{\beta(X)^{-1}}]{\beta^{-1}(t)=\beta(1-t) \text{ Transformation of integral}} (-1)^{\text{wt}(\mathbf{l})} \zeta_{RS}(\mathbf{k}, \bar{\mathbf{l}}; T) + P_{r,s}(T)$$

Point 2:



$$\begin{aligned} & (-1)^{\text{wt}(\mathbf{l})} I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) I_{\beta^{-1}}(\beta^{-1}(1-X); \overline{\text{bs}(\mathbf{l})}; \beta^{-1}(1-\epsilon)) \\ &= (-1)^{\text{wt}(\mathbf{l})} I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) \left( I_\beta(\beta(X); \overline{\text{bs}(\mathbf{l})}; \beta(1-\epsilon)) \right. \\ & \quad \left. + \sum_{i=1}^{i_2+j_2+1} I_\beta(\beta(X); \overline{\text{bs}(\mathbf{l})}^{i+1}; \beta(1-\epsilon)) I_{\beta^{-1}}(\beta^{-1}(\epsilon); \overline{\text{bs}(\mathbf{l})}_i; \beta^{-1}(1-\epsilon)) \right), \end{aligned}$$

where  $\mathbf{a}_i := (a_1, \dots, a_i)$  and  $\mathbf{a}^{i+1} := (a_{i+1}, \dots, a_m)$  for  $\mathbf{a} = (a_1, \dots, a_m) \in \{0, 1\}^m$ .

$$I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}); \beta(X)) \times I_\beta(\beta(\epsilon); \text{bs}(\mathbf{l}); \beta(X)) \xrightarrow[\int_{\epsilon < X < 1-\epsilon} I_{r,s}(X) \frac{d\beta(X)}{\beta(X)-1}]{\beta^{-1}(t)=\beta(1-t) \text{ Transformation of integral}} (-1)^{\text{wt}(\mathbf{l})} \zeta_{RS}(\mathbf{k}, \bar{\mathbf{l}}; T) + P_{r,s}(T).$$

$$(-1)^{\text{wt}(\mathbf{l})} \left( I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}), \mathbf{1}, \overline{\text{bs}(\mathbf{l})}; \beta(1-\epsilon)) + \sum_{i=1}^{i_2+j_2+1} I_\beta(\beta(\epsilon); \text{bs}(\mathbf{k}), \mathbf{1}; \overline{\text{bs}(\mathbf{l})}^{i+1}; \beta(1-\epsilon)) I_{\beta-1}(\beta^{-1}(\epsilon); \overline{\text{bs}(\mathbf{l})}_i; \beta^{-1}(1-\epsilon)) \right).$$

By  $\beta \rightsquigarrow \beta_n$  and  $2\pi i n \rightsquigarrow T$ , we have

$$(-1)^{\text{wt}(\mathbf{l})} \zeta_{RS}(\mathbf{k}, \bar{\mathbf{l}}; T) + \sum_{h=0}^{i_2} \sum_{m=0}^{l_h-1} \frac{(-1)^{k+h+1}}{T} L(e_{k_0} \cdots e_{k_{i_1}} e_{l_{i_2}} \cdots e_{l_{h-m}}; T) L(e_{l_0-1} \cdots e_{l_{h-1}} e_{m+1}; T).$$

These calculation in terms of  $I_{r,s}(X)$ :

$$\begin{aligned}
 & \sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} (-1)^{i_2+j_2+1} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \left( \zeta_{RS}(\mathbf{k}, \mathbf{l}; T) \right. \\
 & + \left. \sum_{h=0}^{i_2} \sum_{m=0}^{l_h-1} \frac{(-1)^{k+h+1}}{T} L(e_{k_0} \cdots e_{k_{i_1}} e_{l_{i_2}} \cdots e_{l_{h-m}}; T) L(e_{l_{0-1}} \cdots e_{l_{h-1}} e_{m+1}; T) \right) \\
 & = \sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1}) (\lambda_1 + \lambda_2)^{i_2} (\xi_1 + \xi_2)^{j_2} \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{RS}(\mathbf{k}, \mathbf{l}; T).
 \end{aligned}$$

Since  $\begin{cases} \zeta_{RS}(\mathbf{k}; 0) = \zeta_S^{\text{III}}(\mathbf{k}), \\ L(w; T) \in T \mathcal{Z}[T], \end{cases}$  green terms vanish and we finally have WSFs for  $\zeta_S^{\text{III}}(\mathbf{k})$ .

□

Weighted sum formulas for  $\zeta_{\mathcal{F}}$  For parameters  $\lambda_1, \lambda_2, \xi_1, \xi_2$  and  $r, s \in \mathbb{Z}_{\geq 0}$ ,

$$\sum_{\substack{i_1+i_2=r \\ j_1+j_2=s}} \left( (-1)^{i_2+j_2} \lambda_1^{i_1} \lambda_2^{i_2} \xi_1^{j_1} \xi_2^{j_2} + (\lambda_1^{i_1} \xi_1^{j_1} + \lambda_2^{i_1} \xi_2^{j_1}) (\lambda_1 + \lambda_2)^{i_2} (\xi_1 + \xi_2)^{j_2} \right) \\ \times \sum_{\substack{\mathbf{k} \in I(i_1+j_1+1, i_1+1) \\ \mathbf{l} \in I(i_2+j_2+1, i_2+1)}} \zeta_{\mathcal{F}}(\mathbf{k}, \mathbf{l}) = 0.$$

Remark. WSFs for  $\mathcal{F} = \mathcal{S}$  hold **without mod  $\zeta(2)$** .

Thank you for your kind attention!!