

# A congruence between symmetric multiple zeta-star values and multiple zeta-star values

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1 Introduction

2 Main Theorem and Corollaries

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# Multiple zeta(-star) values (MZ(S)V)

## Definition 1 (MZ(S)V)

For  $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{>0})^r$  with  $k_r > 1$ ,

$$\zeta(k_1, \dots, k_r) = \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

$$\zeta^*(k_1, \dots, k_r) = \sum_{0 < m_1 \leq \dots \leq m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}.$$

For an index  $\mathbf{k} = (k_1, \dots, k_r)$ ,

- the weight  $\text{wt}(\mathbf{k}) := k_1 + \dots + k_r$ ,
- the depth  $\text{dep}(\mathbf{k}) := r$ ,
- the height  $\text{ht}(\mathbf{k}) := \#\{i \mid k_i > 1\}$ .

## Several relations for MZSVs

$$I_0(k, r, s) = \{\mathbf{k} : \text{admissible} \mid \text{wt}(\mathbf{k}) = k, \text{dep}(\mathbf{k}) = r, \text{ht}(\mathbf{k}) = s\},$$

$$I_0(k, *, s) = \bigcup_r I_0(k, r, s), \quad I_0(k, r, *) = \bigcup_s I_0(k, r, s).$$

- **Aoki–Ohno's relation** [Aoki–Ohno (2005)]

$$\sum_{\mathbf{k} \in I_0(k, *, s)} \zeta^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \zeta(k) \quad (k, s \in \mathbb{Z}_{>0}).$$

- **Sum formula** [Granville (1996), Zagier (1995)]

$$\sum_{\mathbf{k} \in I_0(k, r, *)} \zeta^*(\mathbf{k}) = \binom{k-1}{r-1} \zeta(k) \quad (k, r \in \mathbb{Z}_{>0}).$$

- **Generalized height-one duality** [Li (2012)]

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta^*(\mathbf{k}) - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta^*(\mathbf{k}) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots],$$

$$(m, n \in \mathbb{Z}_{>0}).$$

# Symmetric MZ(S)V (SMZ(S)V)

## Definition 2 (SMZ(S)V)

For  $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{>0})^r$ ,  $M \in \mathbb{Z}_{>0}$ ,

$$\zeta_M^*(\mathbf{k}) = \sum_{0 < m_1 \leq \dots \leq m_r < M} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

$$\zeta_{\mathcal{S}, M}^*(\mathbf{k}) = \sum_{i=0}^r (-1)^{k_{i+1} + \dots + k_r} \zeta_M^*(k_1, \dots, k_i) \zeta_M^*(k_r, \dots, k_{i+1}).$$

We define symmetric multiple zeta(-star) values by

$$\zeta_{\mathcal{S}}^*(\mathbf{k}) := \lim_{M \rightarrow \infty} \zeta_{\mathcal{S}, M}^*(\mathbf{k}) \pmod{\zeta(2)}.$$

# Finite MZ(S)V (FMZ(S)V)

- $\mathcal{A} := (\prod_p \mathbb{F}_p) / (\bigoplus_p \mathbb{F}_p)$  where  $p$  runs over all primes.
- For an index  $\mathbf{k}$ , finite multiple zeta(-star) values are defined by

$$\zeta_{\mathcal{A}}^*(\mathbf{k}) := \left( \zeta_p^*(\mathbf{k}) \bmod p \right)_p \in \mathcal{A}.$$

- Kaneko–Zagier conjectured that

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{A}} & \xrightarrow{?} & \mathcal{Z} / \zeta(2)\mathcal{Z} \\ \Psi & & \Psi \\ \zeta_{\mathcal{A}}(\mathbf{k}) & \mapsto & \zeta_{\mathcal{S}}(\mathbf{k}) \end{array}$$

where  $\mathcal{Z}$  is the  $\mathbb{Q}$ -linear space spanned by 1 and all MZVs and  $\mathcal{Z}_{\mathcal{A}}$  is the  $\mathbb{Q}$ -linear space spanned by 1 and all FMZVs .

## Several relations for SMZSVs and FMZSVs

$$\mathcal{F} \in \{\mathcal{S}, \mathcal{A}\}, \mathfrak{z}_{\mathcal{F}}(k) = \begin{cases} \zeta(k) & (\mathcal{F} = \mathcal{S}), \\ \left(\frac{B_{p-k}}{k}\right)_p & (\mathcal{F} = \mathcal{A}), \end{cases} \quad (B_k: \text{Bernoulli number}).$$

- **Aoki–Ohno's relation** [ $\mathcal{A}$ : Kaneko–Oyama–Saito (2018),  $\mathcal{S}$ : ?]

$$\sum_{\mathbf{k} \in I_0(k, *, s)} \zeta_{\mathcal{F}}^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \mathfrak{z}_{\mathcal{F}}(k) \quad (k, s \in \mathbb{Z}_{>0}).$$

- **Sum formula** [ $\mathcal{A}$ : Saito–Wakabayashi (2015),  $\mathcal{S}$ : Murahara (2016)]

$$\sum_{\mathbf{k} \in I_0(k, r, *)} \zeta_{\mathcal{F}}^*(\mathbf{k}) = \left( \binom{k-1}{r-1} + (-1)^r \right) \mathfrak{z}_{\mathcal{F}}(k) \quad (k, r \in \mathbb{Z}_{>0}).$$

- **Generalized height-one duality** [ $\mathcal{F}$ : Sakurada (2019)]

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_{\mathcal{F}}^*(\mathbf{k}) - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_{\mathcal{F}}^*(\mathbf{k}) = 0 \quad (m, n, s \in \mathbb{Z}_{>0}).$$

	$\zeta^*$	$\zeta_A^*$	$\zeta_S^*$
Aoki–Ohno's relation	○	○	
Sum formula	○	○	○
Generalized height-one duality	○	○	○

	$\zeta^*$	$\zeta_A^*$	$\zeta_S^*$
Aoki–Ohno's relation	○	○	○
Sum formula	○	○	○
Generalized height-one duality	○	○	○

We prove these relations for  $\zeta_S^*(\mathbf{k})$  at once.

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## Theorem 3 (Main Theorem)

For  $k, r, s \in \mathbb{Z}_{>0}$ ,

$$\sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_S^*(\mathbf{k}) \equiv \sum_{\mathbf{k} \in I_0(k, r, s)} \zeta^*(\mathbf{k}) + P_{k, r, s} \pmod{\zeta(2)},$$

where  $P_{k, r, s} \in \mathbb{Q}[\zeta(3), \zeta(5), \zeta(7), \dots]$  by

$$\begin{aligned} & \frac{1}{\alpha\beta} \left( \frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\beta)\Gamma(1-y+\beta)} - 1 \right) \\ & \equiv \sum_{k, r, s \geq 0} P_{k, r, s} x^{k-r-s} y^{r-s} z^{2s-2} \in \mathcal{Z}[[x, y, z]] \pmod{\zeta(2)}, \end{aligned}$$

with  $\alpha + \beta = x + y$ ,  $\alpha\beta = xy - z^2$ .

Remark  $\Gamma(1+x) = \exp\left(-\gamma x + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} x^n\right) \quad (|x| < 1).$

# Corollaries of Main Theorem

- **Aoki–Ohno's relation**  $\dots$  If  $x = y \Rightarrow P_{k,r,s} \equiv 0 \pmod{\zeta(2)}$ .

$$\sum_{\mathbf{k} \in I_0(k,*,s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) \equiv \sum_{\mathbf{k} \in I_0(k,*,s)} \zeta^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1-2^{-k}) \zeta(k).$$

- **Sum formula**  $\dots$  If  $z^2 = xy \Rightarrow P_{k,r,s} \equiv (-1)^r \zeta(k) \pmod{\zeta(2)}$ .

$$\begin{aligned} \sum_{\mathbf{k} \in I_0(k,r,*)} \zeta_{\mathcal{S}}^*(\mathbf{k}) &\equiv \sum_{\mathbf{k} \in I_0(k,r,*)} \zeta^*(\mathbf{k}) + (-1)^r \zeta(k) \\ &= \left( \binom{k-1}{r-1} + (-1)^r \right) \zeta(k). \end{aligned}$$

- **Generalized height-one duality**  $\dots$   $\mathcal{S}$  analogue of the generalized height-one duality vanishes because of  $P_{k,r,s}$ .

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# Sketch of the proof of Main Theorem

## Definition 4

For  $\mathbf{k} = (k_1, \dots, k_r)$ ,  $M \in \mathbb{Z}_{>0}$ , we define

$$\mathcal{L}_M^*(\mathbf{k}; t) := \sum_{\substack{m_1 \prec \dots \prec m_r \\ 0 < |m_1|, \dots, |m_{r-1}| \leq M \\ -M \leq m_r < 0}} \frac{t^{-m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}[t],$$

where “ $\prec$ ”:  $1 \prec 2 \prec \dots \prec (\infty = -\infty) \prec \dots \prec -2 \prec -1$ .

Remark  $\mathcal{L}_M^*(\mathbf{k}; 0) = 0$ ,  $\mathcal{L}_M^*(\mathbf{k}; 1) = \zeta_{\mathcal{S}, M}^*(\mathbf{k}) - \zeta_M^*(\mathbf{k})$ .

And we consider a differential equation of the following generating function:

$$\begin{cases} \Delta_{0, M}^*(t) = \sum_{k, r, s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k, r, s)} \mathcal{L}_M^*(\mathbf{k}; t) \right) x^{k-r-s} y^{r-s} z^{2s-2}, \\ \Delta_{0, M}^*(0) = 0. \end{cases}$$

Differential equation of  $\Delta_{0,M}^*(t)$

$$t(1-t) \frac{d^2 \Delta_{0,M}^*(t)}{dt^2} + \left( (1-t)(1+x) - ty \right) \frac{d\Delta_{0,M}^*(t)}{dt} - (xy - z^2) \Delta_{0,M}^*(t) = \Phi_{\mathcal{S},M}^* - t^M \Phi_M^*$$

Unique solution:  $\Delta_{0,M}^*(t) = \Phi_{\mathcal{S},M}^* u_1(t) - \Phi_M^* u_{2,M}(t).$

$$\Phi_{\bullet,M}^* = 1 + \sum_{\substack{k,r,s \geq 0 \\ (k,r,s) \neq (0,0,0)}} \left( \sum_{\mathbf{k} \in I(k,r,s)} \zeta_{\bullet,M}^*(\mathbf{k}) \right) x^{k-r-s} y^{r-s} z^{2s} \quad (\bullet \in \{\emptyset, \mathcal{S}\}),$$

$$u_1(t) = \frac{1}{\alpha\beta} \left( {}_2F_1 \left( \begin{matrix} \alpha, \beta \\ x+1 \end{matrix}; t \right) - 1 \right), \quad \begin{cases} \alpha + \beta = x + y, \\ \alpha\beta = xy - z^2, \end{cases}$$

$$u_{2,M}(t) = \frac{t^M}{(\alpha + M)(\beta + M)} \left( {}_3F_2 \left( \begin{matrix} \alpha + M, \beta + M, 1 \\ x + M + 1, M + 1 \end{matrix}; t \right) - 1 \right).$$

Put  $t = 1$  and take the limit  $M \rightarrow \infty$ ,

$$\Delta_{0,M}^*(1) = \underbrace{\Phi_{\mathcal{S},M}^* u_1(1)}_{\rightarrow \Phi_{\mathcal{S}}^* u_1(1) \equiv u_1(1)} - \underbrace{\Phi_M^* u_{2,M}(1)}_{\rightarrow 0}.$$

Put  $t = 1$  and take the limit  $M \rightarrow \infty$ ,

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$$f = \sum_{a,b,c \geq 0} X(a,b,c) x^a y^b z^c \implies \text{Coeff}(f; x^a y^b z^c) := X(a,b,c).$$

$$\left\{ \begin{aligned} \text{Coeff}(\Phi_M^*; x^{k-r-s} y^{r-s} z^{2s}) &= \sum_{\mathbf{k} \in I(k,r,s)} \zeta_M^*(k_1, \dots, k_r) \\ &\leq \left( 1 + \int_1^M \frac{1}{m} dm \right)^r = O(\log^r M), \\ \text{Coeff}(u_{2,M}(1); x^{k-r-s} y^{r-s} z^{2s-2}) &= O\left(\frac{1}{M}\right), \end{aligned} \right.$$

$$\implies \text{Coeff}(\Phi_M^* u_{2,M}(1); x^{k-r-s} y^{r-s} z^{2s}) = O\left(\frac{\log^r M}{M}\right) \rightarrow 0 \quad (M \rightarrow \infty).$$

Put  $t = 1$  and take the limit  $M \rightarrow \infty$ ,

$$\Delta_{0,M}^*(1) = \underbrace{\Phi_{S,M}^* u_1(1)}_{\rightarrow \Phi_S^* u_1(1) \equiv u_1(1)} - \underbrace{\Phi_M^* u_{2,M}(1)}_{\rightarrow 0}.$$

$$\lim_{M \rightarrow \infty} \Delta_{0,M}^*(1) = \sum_{k,r,s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k,r,s)} \left( \zeta_S^*(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) \right) x^{k-r-s} y^{r-s} z^{2s-2},$$

$$\lim_{M \rightarrow \infty} \Phi_{S,M}^* u_1(1) = \left( 1 + \sum_{\substack{k,r,s \geq 0 \\ (k,r,s) \neq 0}} \underbrace{\left( \sum_{\mathbf{k} \in I(k,r,s)} \zeta_S^*(\mathbf{k}) \right)}_{\equiv 0} x^{k-r-s} y^{r-s} z^{2s} \right) u_1(1)$$

$$\equiv \frac{1}{\alpha\beta} \left( \frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\beta)\Gamma(1-y+\beta)} - 1 \right) \text{ mod } \zeta(2)$$

$$\equiv \sum_{k,r,s \geq 0} P_{k,r,s} x^{k-r-s} y^{r-s} z^{2s-2} \text{ mod } \zeta(2).$$

$$(\alpha + \beta = x + y, \alpha\beta = xy - z^2)$$

Comparing these equations, we obtain Main Theorem. □

Put  $t = 1$  and take the limit  $M \rightarrow \infty$ ,

$$\Delta_{0,M}^*(1) = \underbrace{\Phi_{\mathcal{S},M}^* u_1(1)}_{\rightarrow \Phi_{\mathcal{S}}^* u_1(1) \equiv u_1(1)} - \underbrace{\Phi_M^* u_{2,M}(1)}_{\rightarrow 0}.$$

$$\lim_{M \rightarrow \infty} \Delta_{0,M}^*(1) = \sum_{k,r,s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k,r,s)} \left( \zeta_{\mathcal{S}}^*(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) \right) x^{k-r-s} y^{r-s} z^{2s-2},$$

$$\lim_{M \rightarrow \infty} \Phi_{\mathcal{S},M}^* u_1(1) = \left( 1 + \sum_{\substack{k,r,s \geq 0 \\ (k,r,s) \neq 0}} \underbrace{\left( \sum_{\mathbf{k} \in I(k,r,s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) \right)}_{\equiv 0} x^{k-r-s} y^{r-s} z^{2s} \right) u_1(1)$$

$$\equiv \frac{1}{\alpha\beta} \left( \frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\beta)\Gamma(1-y+\beta)} - 1 \right) \text{ mod } \zeta(2)$$

$$\equiv \sum_{k,r,s \geq 0} P_{k,r,s} x^{k-r-s} y^{r-s} z^{2s-2} \text{ mod } \zeta(2).$$

$$(\alpha + \beta = x + y, \alpha\beta = xy - z^2)$$

Comparing these equations, we obtain Main Theorem. □

# Summary

Main Theorem

$$\sum_{\mathbf{k} \in I_0(k,r,s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) \equiv \sum_{\mathbf{k} \in I_0(k,r,s)} \zeta^*(\mathbf{k}) + P_{k,r,s} \pmod{\zeta(2)}.$$

Set  $\bullet \in \{\emptyset, \mathcal{S}\}$ ,  $Q \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots]$ ,

$$\text{A.O. : } \sum_{\mathbf{k} \in I_0(k,*,s)} \zeta_{\bullet}^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \zeta(k),$$

$$\text{S.F. : } \sum_{\mathbf{k} \in I_0(k,r,*)} \zeta_{\bullet}^*(\mathbf{k}) = \left( \binom{k-1}{r-1} + \delta_{\bullet, \mathcal{S}} (-1)^r \right) \zeta(k),$$

$$\text{G.H.-O.D. : } (-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_{\bullet}^*(\mathbf{k}) - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_{\bullet}^*(\mathbf{k}) = \delta_{\bullet, \emptyset} Q.$$

Remark 
$$\sum_{\mathbf{k} \in I_0(k,r,s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) = (-1)^{k-1} \sum_{\mathbf{k} \in I_0(k,r,s)} (-1)^r \zeta_{\mathcal{S}}(\mathbf{k}).$$

Thank you for your attention!