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対称多重ゼータスター値と多重ゼータスター値間のある合同式について

A Congruence Between Symmetric Multiple Zeta-Star Values and Multiple Zeta-Star Values

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## **Abstract**

In this thesis, we prove Aoki–Ohno’s relation for symmetric multiple zeta-star values and give new proofs of a sum formula and a generalization of the height-one duality for these values. For this purpose, we introduce truncated polylogarithm-like sums and prove a congruence between symmetric multiple zeta-star values and multiple zeta-star values.

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# Chapter 1

## Introduction

Multiple zeta values are generalizations of special values of the Riemann zeta function which have been studied actively since the late 1980s. It is known that the  $\mathbb{Q}$ -linear space  $\mathcal{Z}$  spanned by all multiple zeta values has a structure of  $\mathbb{Q}$ -algebra. Zagier conjectured that there are many  $\mathbb{Q}$ -linear relations among these values. Thus it is important to study what relations hold among them.

In recent years, besides ordinary multiple zeta values, two variants of multiple zeta values have been studied extensively: finite multiple zeta values and symmetric multiple zeta values. Although Hoffman and Zhao considered independently a truncated version of multiple zeta values for each prime  $p$ , which are called multiple harmonic sums, Zagier suggested around 2010 that we should consider a collection of mod  $p$  multiple harmonic sums for all prime numbers as an element of  $\mathbb{Q}$ -algebra  $\mathcal{A} = (\prod_p \mathbb{Z}/p\mathbb{Z})/(\bigoplus_p \mathbb{Z}/p\mathbb{Z})$ . Kaneko and Zagier called these elements finite multiple zeta values and found that the  $\mathbb{Q}$ -linear space generated by all finite multiple zeta values are equipped with a  $\mathbb{Q}$ -algebra structure. As in the case of ordinary multiple zeta values, they also proposed that there are many  $\mathbb{Q}$ -linear relations among finite multiple zeta values and it is significant to research these relations. Further, by defining symmetric multiple zeta values as a real counterpart of finite multiple zeta values, Kaneko and Zagier predicted a one-to-one correspondence between these two kinds of values, which is called the Kaneko–Zagier conjecture. This conjecture implies that any relations for the finite multiple zeta values take the same form as those for symmetric multiple zeta values, and vice versa.

In this thesis, our main purpose is to prove a congruence between symmetric multiple zeta-star values and multiple zeta-star values and to give proofs of three relations for symmetric multiple zeta values which include the relation not yet proved. These relations support the Kaneko–Zagier conjecture.

In the following, we state the structure of this thesis. First, we review some basic properties of multiple zeta values in Chapter 2. Secondly, in Chapter 3, we state the definitions of finite multiple zeta values and symmetric multiple zeta values and explain the Kaneko–Zagier conjecture. We

also introduce several relations for finite and symmetric multiple zeta values. Lastly, in Chapter 4, we prove our main theorem, that is, a congruence between symmetric multiple zeta-star values and multiple zeta-star values and give proofs for three relations for symmetric multiple zeta values including one which was conjectured. Chapter 4 is due to [6].

# Chapter 2

## Multiple zeta values

In this chapter, we define the multiple zeta(-star) values (MZ(S)V for short) and state fundamental facts about these values.

### 2.1 Notation for MZVs

An index is a sequence of positive integers and written as  $\mathbf{k} = (k_1, \dots, k_r)$ . In particular, an index  $\mathbf{k} = (k_1, \dots, k_r)$  is called an admissible index if  $k_r > 1$ . For an index  $\mathbf{k}$ , we set

$$\text{wt}(\mathbf{k}) = k_1 + \dots + k_r, \quad \text{dep}(\mathbf{k}) = r, \quad \text{ht}(\mathbf{k}) = \#\{i \mid k_i > 1\},$$

which are said to be the weight, the depth and the height of  $\mathbf{k}$ , respectively.

**Definition 2.1.1** (MZV and MZSV). For an admissible index  $\mathbf{k} = (k_1, \dots, k_r)$ , the MZV is defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) := \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

and the MZSV is defined by

$$\zeta^*(\mathbf{k}) = \zeta^*(k_1, \dots, k_r) := \sum_{0 < m_1 \leq \dots \leq m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}.$$

Conventionally, we define  $\zeta(\emptyset) = \zeta^*(\emptyset) = 1$ .

Let  $\mathbf{k}, \mathbf{k}'$  be indices which have the same weight. For these indices, we set a binary relation “ $\preceq$ ” as

$$\mathbf{k}' \preceq \mathbf{k} \stackrel{\text{def.}}{\iff} \mathbf{k}' \text{ is obtained from } \mathbf{k} = (k_1, \dots, k_r) \text{ by replacing some “,”s with “+”s.}$$

By using the relation above, MZSVs are written in terms of MZVs.

**Proposition 2.1.2.** *For an admissible index  $\mathbf{k}$ , we have*

$$\zeta^*(\mathbf{k}) = \sum_{\mathbf{k}' \leq \mathbf{k}} \zeta(\mathbf{k}'). \quad (2.1.1)$$

*Proof.* (2.1.1) follows from dividing each “ $\leq$ ” into “ $<$ ” and “ $=$ ” in the definition of  $\zeta^*(\mathbf{k})$ . For example,

$$\begin{aligned} \zeta^*(1, 2) &= \sum_{0 < m \leq n} \frac{1}{mn^2} \\ &= \left( \sum_{0 < m < n} + \sum_{0 < m = n} \right) \frac{1}{mn^2} \\ &= \zeta(1, 2) + \zeta(3) \\ &= \sum_{\mathbf{k} \leq (1, 2)} \zeta(\mathbf{k}). \end{aligned}$$

□

## 2.2 Zagier’s conjecture

In this section, we state Zagier’s conjecture concerning  $\mathbb{Q}$ -vector spaces spanned by MZVs.

**Definition 2.2.1.** For a non-negative integer  $k$ , the  $\mathbb{Q}$ -vector space  $\mathcal{Z}_k$  is defined by

$$\begin{aligned} \mathcal{Z}_0 &:= \mathbb{Q}, \quad \mathcal{Z}_1 := \{0\}, \\ \mathcal{Z}_k &:= \sum_{\substack{\mathbf{k}: \text{admissible} \\ \text{wt}(\mathbf{k})=k}} \mathbb{Q} \cdot \zeta(\mathbf{k}) \quad (k \geq 2), \end{aligned}$$

and  $\mathcal{Z}$  is defined by

$$\mathcal{Z} := \sum_{k=0}^{\infty} \mathcal{Z}_k.$$

*Remark.* The  $\mathcal{Z}_k$  is the space generated by all MZVs whose weight is  $k$ . It is expected that this is a direct sum. However, it is not yet proved by the same reason as in Zagier’s conjecture stated below.

In [27], Zagier conjectured that the dimension of  $\mathcal{Z}_k$  is equal to  $d_k$  defined by the recurrence  $d_k = d_{k-2} + d_{k-3}$  ( $k \geq 3$ ) with  $d_0 := 1, d_1 := 0, d_2 := 1$ . About this conjecture, the following result is known.

**Theorem 2.2.2** (Goncharov [7], Terasoma [23], Deligne–Goncharov [5]).

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k \quad (k \geq 0).$$

It is very difficult to prove  $\dim_{\mathbb{Q}} \mathcal{Z}_k \geq d_k$  because we need to prove that MZVs are linearly independent. Thus our motivation to study MZVs is to investigate what sorts of relations are needed to reduce the number of generators to the upper bound. As shown in the table below, the dimension  $d_k$  of  $\mathcal{Z}_k$  is much smaller, than the total number of index sets of weight  $k (= 2^{k-2})$ . Therefore, it is shown that there are at least  $2^{k-2} - d_k$  linear relations among the MZVs of the same weight.

| $k$       | 0 | 1 | 2 | 3 | 4 | 5 | 6  | 7  | 8  | 9   | 10  | 11  | 12   | 13   | 14   | 15   |
|-----------|---|---|---|---|---|---|----|----|----|-----|-----|-----|------|------|------|------|
| $d_k$     | 1 | 0 | 1 | 1 | 1 | 2 | 2  | 3  | 4  | 5   | 7   | 9   | 12   | 16   | 21   | 28   |
| $2^{k-2}$ | – | – | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2018 | 4096 | 8192 |

Next, we show that the space  $\mathcal{Z}$  has a structure of  $\mathbb{Q}$ -algebra.

**Proposition 2.2.3.** *The space  $\mathcal{Z}$  is a  $\mathbb{Q}$ -algebra, and the multiplication respects weights i.e.,  $\mathcal{Z}_k \cdot \mathcal{Z}_l \subset \mathcal{Z}_{k+l}$ .*

*Proof.* To prove this proposition, we introduce the truncated MZVs. For an index  $\mathbf{k} = (k_1, \dots, k_r)$  and a positive integer  $M$ , we set

$$\zeta_M(\mathbf{k}) = \zeta_M(k_1, \dots, k_r) := \sum_{0 < m_1 < \dots < m_r \leq M} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

It is clear that if  $\mathbf{k}$  is admissible, then the limit  $\lim_{M \rightarrow \infty} \zeta_M(\mathbf{k}) = \zeta(\mathbf{k})$  holds.

For any positive integer  $M$  and two admissible indices  $\mathbf{k} = (k_1, \dots, k_r)$  and  $\mathbf{l} = (l_1, \dots, l_s)$ , we show the product  $\zeta_M(\mathbf{k})\zeta_M(\mathbf{l})$  is expressed as the sum of several truncated MZVs whose weight is  $\text{wt}(\mathbf{k}) + \text{wt}(\mathbf{l})$  by induction on  $r + s$ .

We set  $r \leq s$  without loss of generality. If  $r + s = 2$  namely,  $r = s = 1$ , then we have

$$\begin{aligned} \zeta_M(k)\zeta_M(l) &= \left( \sum_{0 < m \leq M} \frac{1}{m^k} \right) \left( \sum_{0 < n \leq M} \frac{1}{n^l} \right) \\ &= \left( \sum_{0 < m < n \leq M} + \sum_{0 < n < m \leq M} + \sum_{0 < m = n \leq M} \right) \frac{1}{m^k n^l} \\ &= \zeta_M(k, l) + \zeta_M(l, k) + \zeta_M(k + l). \end{aligned}$$

Assume that the assertion is true for less than  $r + s$ . We obtain

$$\begin{aligned}
\zeta_M(\mathbf{k})\zeta_M(\mathbf{l}) &= \left( \sum_{0 < m_1 < \dots < m_r \leq M} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \right) \left( \sum_{0 < n_1 < \dots < n_s \leq M} \frac{1}{n_1^{l_1} \dots n_s^{l_s}} \right) \\
&= \sum_{\substack{r \leq m_r < n_s \leq M \\ s \leq n_s}} \left( \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \right) \left( \sum_{0 < n_1 < \dots < n_s} \frac{1}{n_1^{l_1} \dots n_s^{l_s}} \right) \\
&\quad + \sum_{s \leq n_s < m_r \leq M} \left( \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \right) \left( \sum_{0 < n_1 < \dots < n_s} \frac{1}{n_1^{l_1} \dots n_s^{l_s}} \right) \\
&\quad + \sum_{s \leq n_s = m_r \leq M} \left( \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \right) \left( \sum_{0 < n_1 < \dots < n_s} \frac{1}{n_1^{l_1} \dots n_s^{l_s}} \right).
\end{aligned}$$

Here, the second term is equal to

$$\begin{aligned}
\sum_{s < m_r \leq M} \frac{1}{m_r^{k_r}} \left( \sum_{0 < m_1 < \dots < m_r} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \right) \left( \sum_{0 < n_1 < \dots < n_s < m_r} \frac{1}{n_1^{l_1} \dots n_s^{l_s}} \right) \\
&= \sum_{s < m_r \leq M} \frac{1}{m_r^{k_r}} \zeta_{m_r-1}(k_1, \dots, k_{r-1}) \zeta_{m_r-1}(\mathbf{l}) \\
&= \sum_{\mathbf{m}} C_{\mathbf{m}} \zeta_M(\mathbf{m}, k_r),
\end{aligned}$$

where the last sum runs over all indices  $\mathbf{m}$  whose weight is  $\text{wt}(\mathbf{k}) + \text{wt}(\mathbf{l}) - k_r$ , the coefficients  $C_{\mathbf{m}} \in \mathbb{Z}_{\geq 0}$  are some constants and we used the assumption at the last equality. Similarly, it is easily checked that the other terms are also expressed as a sum of truncated MZVs. Lastly, taking the limit  $M \rightarrow \infty$  leads to the claim.  $\square$

*Remark.* By Proposition 2.2.3, we find that a product of two MZVs (resp. truncated MZVs) is written in terms of a combination of MZVs (resp. truncated MZVs). This multiplication is called the harmonic product.

## 2.3 The iterated integral expression of MZVs

Although MZVs are defined in terms of series, these values also have integral expressions. In this section, we introduce iterated integrals and give integral expressions of MZVs. For this purpose,

we set the iterated integral  $I(\epsilon_1, \dots, \epsilon_k)$  as follows:

$$I(\epsilon_1, \dots, \epsilon_k) := \int_{0 < t_1 < \dots < t_k < 1} \dots \int A_{\epsilon_1}(t_1) dt_1 \dots A_{\epsilon_k}(t_k) dt_k, \quad (2.3.1)$$

where  $\epsilon_j \in \{0, 1\}$  for  $1 \leq j \leq k$ , and

$$A_0(t) = \frac{1}{t}, \quad A_1(t) = \frac{1}{1-t}.$$

For the convergence of the integral, we set  $\epsilon_1 = 1$  and  $\epsilon_k = 0$ . This integral plays an important role to lead to the integral expressions of MZVs.

**Theorem 2.3.1** (Integral expressions of MZVs). *For an admissible index  $(k_1, \dots, k_r)$ , we have*

$$\zeta(k_1, \dots, k_r) = I(1, \underbrace{0, \dots, 0}_{k_1-1}, 1, \dots, 1, \underbrace{0, \dots, 0}_{k_r-1}).$$

Theorem 2.3.1 is obtained by expanding  $1/(1-t_i)$  to  $\sum_{n=1}^{\infty} t_i^n$  and repeatedly applying termwise integral for this series. Instead, in this thesis, we introduce the multi-polylogarithms, and prove Theorem 2.3.1 as a special case of the integral expressions of the multi-polylogarithms.

**Definition 2.3.2** (Multi-polylogarithm). For an index  $\mathbf{k} = (k_1, \dots, k_r)$ , the multi-polylogarithm  $\text{Li}_{\mathbf{k}}(z)$  is defined by

$$\text{Li}_{\mathbf{k}}(z) = \text{Li}_{k_1, \dots, k_r}(z) := \sum_{0 < m_1 < \dots < m_r} \frac{z^{m_r}}{m_1^{k_1} \dots m_r^{k_r}} \quad (|z| < 1).$$

Conventionally, we define  $\text{Li}_{\emptyset}(z) := 1$ .

*Remark.* The multi-polylogarithm is holomorphic on  $|z| < 1$ . Moreover, if the index  $\mathbf{k}$  is admissible, then this function converges as  $z \rightarrow 1$  and the limit is given by

$$\lim_{z \rightarrow 1} \text{Li}_{\mathbf{k}}(z) = \zeta(\mathbf{k}).$$

If  $r = 1$ , then  $\text{Li}_k(z) = \sum_{m=1}^{\infty} z^m / m^k$  is called the polylogarithm. In particular, the polylogarithm with  $k = 1$  is given by

$$\text{Li}_1(z) = -\log(1-z) = \int_0^z \frac{1}{1-t} dt. \quad (2.3.2)$$

**Lemma 2.3.3.** For an index  $(k_1, \dots, k_{r-1}, k_r)$  and  $|z| < 1$ , we have

$$\frac{d}{dz} \text{Li}_{k_1, \dots, k_{r-1}, k_r}(z) = \begin{cases} \frac{1}{z} \text{Li}_{k_1, \dots, k_{r-1}, k_{r-1}}(z) & (k_r > 1), \\ \frac{1}{1-z} \text{Li}_{k_1, \dots, k_{r-1}}(z) & (k_r = 1). \end{cases}$$

*Proof.* The first equality is obtained by termwise differentiation.

If  $k_r = 1$ , then we have

$$\begin{aligned} \frac{d}{dz} \text{Li}_{k_1, \dots, k_{r-1}, 1}(z) &= \sum_{0 < m_1 < \dots < m_r} \frac{z^{m_r-1}}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \\ &= \sum_{0 < m_1 < \dots < m_{r-1}} \left( \sum_{m_r=m_{r-1}+1}^{\infty} z^{m_r-1} \right) \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \\ &= \frac{1}{1-z} \sum_{0 < m_1 < \dots < m_{r-1}} \frac{z^{m_r-1}}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}}, \end{aligned}$$

which is equal to the second equality and completes the proof.  $\square$

Hereafter, in this chapter, we abbreviate the iterated integral with differential forms  $\omega_i(t)$  as

$$\int_0^z \omega_1(t) \circ \omega_2(t) \circ \dots \circ \omega_k(t) := \int_0^z \omega_k(t_k) \int_0^{t_k} \omega_{k-1}(t_{k-1}) \dots \int_0^{t_2} \omega_1(t_1).$$

By the expression above, the multi-polylogarithm is written as follows.

**Proposition 2.3.4.** We have

$$\text{Li}_{k_1, k_2, \dots, k_r}(z) = \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1-1} \circ \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_2-1} \circ \dots \circ \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_r-1}. \quad (2.3.3)$$

*Proof.* By applying Lemma 2.3.3 to (2.3.2) repeatedly, we obtain the assertion.  $\square$

Specializing  $z = 1$  in (2.3.3), we have

$$\zeta(k_1, k_2, \dots, k_r) = \int_0^1 \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1-1} \circ \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_2-1} \circ \dots \circ \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_r-1},$$

which gives the proof of Theorem 2.3.1.

The iterated integral  $I(\epsilon_1, \dots, \epsilon_k)$  has the following symmetry.

**Proposition 2.3.5.** For an positive integer  $k$  and  $\epsilon_1, \dots, \epsilon_k \in \{0, 1\}$  ( $\epsilon_1 = 0, \epsilon_k = 1$ ), we have

$$I(\epsilon_1, \dots, \epsilon_k) = I(1 - \epsilon_k, \dots, 1 - \epsilon_1).$$

*Proof.* By exchanging the variables as  $(t_1, \dots, t_k) \rightarrow (1 - s_k, \dots, 1 - s_1)$  in (2.3.1), we have

$$0 < t_1 < \dots < t_k < 1 \iff 0 < s_1 < \dots < s_k < 1,$$

and the absolute value of Jacobian  $|J|$  is given by

$$|J| = \left| \det \begin{pmatrix} \frac{dt_1}{ds_1} & \dots & \frac{dt_1}{ds_k} \\ \vdots & \ddots & \vdots \\ \frac{dt_k}{ds_1} & \dots & \frac{dt_k}{ds_k} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 0 & & -1 \\ & \ddots & \\ -1 & & 0 \end{pmatrix} \right| = 1.$$

Furthermore, by  $A_{\epsilon_i}(1 - s_j) = A_{1 - \epsilon_i}(s_j)$  we have

$$I(\epsilon_1, \dots, \epsilon_k) = \int_{0 < s_1 < \dots < s_k < 1} A_{1 - \epsilon_1}(s_k) ds_k \cdots A_{1 - \epsilon_k}(s_1) ds_1 = I(1 - \epsilon_k, \dots, 1 - \epsilon_1),$$

which completes the proof. □

To apply this symmetry to MZVs, we define the dual index.

**Definition 2.3.6** (Dual index). For an admissible index

$$\mathbf{k} = (\underbrace{1, \dots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \dots, 1}_{a_2 - 1}, b_2 + 1, \dots, \underbrace{1, \dots, 1}_{a_s - 1}, b_s + 1).$$

with  $a_1, \dots, a_s, b_1, \dots, b_s \geq 1$ , we define the dual index of  $\mathbf{k}$  by

$$\mathbf{k}^\dagger = (\underbrace{1, \dots, 1}_{b_s - 1}, a_s + 1, \underbrace{1, \dots, 1}_{b_{s-1} - 1}, a_{s-1} + 1, \dots, \underbrace{1, \dots, 1}_{b_1 - 1}, a_1 + 1).$$

**Example 2.3.7.**  $(1, 2)^\dagger = (3)$ .

From the definition above, the following Theorem holds.

**Theorem 2.3.8** (Duality). For an admissible index  $\mathbf{k}$ , we have

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\dagger).$$

*Proof.* By Lemma 2.3.5, we have

$$\begin{aligned}
\zeta(\mathbf{k}) &= \zeta(\underbrace{1, \dots, 1}_{a_1-1}, \underbrace{b_1 + 1, 1, \dots, 1}_{a_2-1}, \dots, \underbrace{1, \dots, 1}_{a_s-1}, b_s + 1) \\
&= I(\underbrace{1, \dots, 1}_{a_1}, \underbrace{0, \dots, 0}_{b_1}, \dots, \underbrace{1, \dots, 1}_{a_s}, \underbrace{0, \dots, 0}_{b_s}) \\
&= I(\underbrace{1, \dots, 1}_{b_s}, \underbrace{0, \dots, 0}_{a_s}, \dots, \underbrace{1, \dots, 1}_{b_1}, \underbrace{0, \dots, 0}_{a_1}) \\
&= \zeta(\underbrace{1, \dots, 1}_{b_s-1}, \underbrace{a_s + 1, 1, \dots, 1}_{b_{s-1}-1}, \dots, \underbrace{1, \dots, 1}_{b_1-1}, a_1 + 1) \\
&= \zeta(\mathbf{k}^\dagger).
\end{aligned}$$

□

**Theorem 2.3.9** (Shuffle product). *Set  $\omega_1(t) = \omega'_1(t) = dt/(1-t)$  and let each  $\omega_2(t), \dots, \omega_k(t), \omega'_2(t), \dots, \omega'_{k'}(t)$  be either  $dt/t$  or  $dt/(1-t)$ . For  $|z| < 1$ , we have*

$$\begin{aligned}
\left( \int_0^z \omega_1(t) \circ \dots \circ \omega_k(t) \right) \left( \int_0^z \omega'_1(t) \circ \dots \circ \omega'_{k'}(t) \right) \\
= \sum_{(\eta_1(t), \dots, \eta_{k+k'}(t))} \left( \int_0^z \eta_1(t) \circ \dots \circ \eta_{k+k'}(t) \right), \quad (2.3.4)
\end{aligned}$$

where  $(\eta_1(t), \dots, \eta_{k+k'}(t))$  runs over all permutations of  $(\omega_1(t), \dots, \omega_k(t))$  and  $(\omega'_1(t), \dots, \omega'_{k'}(t))$  with its order kept (shuffles of  $(\omega_1(t), \dots, \omega_k(t))$  and  $(\omega'_1(t), \dots, \omega'_{k'}(t))$ ).

*Proof.* We give a proof by induction on  $k + k'$ .

When  $k + k' = 2$  namely,  $k = k' = 1$ , then by Lemma 2.3.3, we have

$$\frac{d}{dz} (\text{Li}_1(z)^2) = 2 \frac{1}{1-z} \text{Li}_1(z) = 2 \frac{d}{dz} \text{Li}_{1,1}(z).$$

Since  $\text{Li}_1(0)^2 = \text{Li}_{1,1}(0) = 0$  holds, we have

$$\text{Li}_1(z)^2 = 2 \text{Li}_{1,1}(z),$$

which implies this case.

Next, let us assume (2.3.4) is true for less than  $k + k'$ . By the definition of iterated integrals and the Leibniz rule, the differential of the left-hand side of (2.3.4) is given by

$$\begin{aligned} & \left( \int_0^z \omega_1(t) \circ \cdots \circ \omega_{k-1}(t) \int_0^z \omega'_1(t) \circ \cdots \circ \omega'_{k'}(t) \right) \omega_k(z) \\ & + \left( \int_0^z \omega_1(t) \circ \cdots \circ \omega_k(t) \int_0^z \omega'_1(t) \circ \cdots \circ \omega'_{k'-1}(t) \right) \omega'_{k'}(z). \end{aligned} \quad (2.3.5)$$

From the assumption of induction, (2.3.5) is equal to

$$\sum_{(\eta_1(t), \dots, \eta_{k+k'}(t))} \left( \int_0^z \eta_1(t) \circ \cdots \circ \eta_{k+k'-1}(t) \right) \eta_{k+k'}(z) = d(\text{R.H.S of (2.3.4)}). \quad (2.3.6)$$

If  $z = 0$ , then both sides of (2.3.6) vanish, so that (2.3.4) holds for  $k + k'$ , which gives the proof.  $\square$

Set  $\omega_k(t) = \omega'_{k'}(t) = 1/t$  as in Theorem 2.3.9. Then both sides of (2.3.4) converge to MZVs. Thus it is clear that the product of two MZVs is the sum of MZVs again. Namely, we gave another proof of Proposition 2.2.3.

**Example 2.3.10.** We calculate the square of  $\zeta(2)$  in two ways:

- Harmonic product

$$\begin{aligned} \zeta(2)\zeta(2) &= \left( \sum_{m_1=1}^{\infty} \frac{1}{m_1^2} \right) \left( \sum_{m_2=1}^{\infty} \frac{1}{m_2^2} \right) \\ &= \left( \sum_{0 < m_1 < m_2} + \sum_{0 < m_2 < m_1} + \sum_{0 < m_1 = m_2} \right) \frac{1}{m_1^2 m_2^2} \\ &= 2\zeta(2, 2) + \zeta(4). \end{aligned}$$

- Shuffle product

$$\begin{aligned}
\zeta(2)\zeta(2) &= \left( \int_0^1 \frac{dt}{1-t} \circ \frac{dt}{t} \right) \left( \int_0^1 \frac{ds}{1-s} \circ \frac{ds}{s} \right) \\
&= \int_0^1 \frac{dt}{1-t} \circ \frac{dt}{t} \circ \frac{ds}{1-s} \circ \frac{ds}{s} + \int_0^1 \frac{dt}{1-t} \circ \frac{ds}{1-s} \circ \frac{dt}{t} \circ \frac{ds}{s} \\
&\quad + \int_0^1 \frac{dt}{1-t} \circ \frac{ds}{1-s} \circ \frac{ds}{s} \circ \frac{dt}{t} + \int_0^1 \frac{ds}{1-s} \circ \frac{dt}{1-t} \circ \frac{ds}{s} \circ \frac{dt}{t} \\
&\quad + \int_0^1 \frac{ds}{1-s} \circ \frac{ds}{s} \circ \frac{dt}{1-t} \circ \frac{dt}{t} + \int_0^1 \frac{ds}{1-s} \circ \frac{dt}{1-t} \circ \frac{dt}{t} \circ \frac{ds}{s} \\
&= 2\zeta(2, 2) + 4\zeta(1, 3).
\end{aligned}$$

Thus we get

$$2\zeta(2, 2) + \zeta(4) = 2\zeta(2, 2) + 4\zeta(1, 3) \quad (2.3.7)$$

and hence

$$\zeta(4) = 4\zeta(1, 3).$$

If we expand the product of two MZVs by shuffle product, then all indices are of the same depth. On the other hand, if we use the harmonic product to expand them, then indices of different depths appear. Therefore, we get non-trivial relations among MZVs by calculating these two products. These relations are said to be finite double shuffle relations. We formulate these relations in an algebraic way in the next section.

## 2.4 Hoffman's algebras

In this section, we consider Hoffman's algebra introduced in [12] and state finite double shuffle relations.

**Definition 2.4.1.** Let  $\mathfrak{H} := \mathbb{Q}\langle x, y \rangle$  be a non-commutative polynomial algebra in two variables and set

$$\mathfrak{H}^1 := \mathbb{Q} + y\mathfrak{H}, \quad \mathfrak{H}^0 := \mathbb{Q} + y\mathfrak{H}x.$$

Note that  $\mathfrak{H}^1$  and  $\mathfrak{H}^0$  are subalgebras of  $\mathfrak{H}$  with  $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$ . Moreover, if we set  $z_k := yx^{k-1}$  for a positive integer  $k$ , then we regard  $\mathfrak{H}^1$  as a non-commutative polynomial algebra generated by all  $z_k$ :

$$\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \dots \rangle.$$

From this fact, any non-constant monomial  $w$  in  $\mathfrak{H}^0$  is uniquely expressed as  $w = z_{k_1} \cdots z_{k_r}$  ( $k_1, \dots, k_{r-1} \geq 1, k_r > 1$ ). In the following, we call a non-constant monomial a word.

**Definition 2.4.2.** For a word  $w = z_{k_1} \cdots z_{k_r} \in \mathfrak{H}^0$ , we define an evaluation map  $Z : \mathfrak{H}^0 \rightarrow \mathbb{R}$  by

$$Z(z_{k_1}, \dots, z_{k_r}) = \zeta(k_1, \dots, k_r), \quad Z(1) = 1.$$

**Example 2.4.3.**  $Z(yx) = \zeta(2)$ ,  $Z(yx^2yx) = \zeta(3, 2)$ .

We remark that the weight (resp. the depth) of an index correspond to the degree of  $x$  (resp.  $y$ ) in a word. Thus we define the weight and the depth for a word in this way.

Next, we define two multiplications which naturally arise from series expressions and integral expressions.

**Definition 2.4.4.** The harmonic product  $*$  on  $\mathfrak{H}^1$  is defined by the  $\mathbb{Q}$ -bilinearity and the following rules:

1.  $w * 1 = 1 * w = w$  for any word  $w \in \mathfrak{H}^1$ ,
2.  $(w_1 z_p * w_2 z_q) = (w_1 * w_2 z_q) z_p + (w_1 z_p * w_2) z_q + (w_1 * w_2) z_{p+q}$   
for any  $w_1, w_2 \in \mathfrak{H}^1$  and positive integers  $p, q$ .

In [12], Hoffman proved the harmonic product is commutative and associative. Hence  $\mathfrak{H}^1$  is a  $\mathbb{Q}$ -algebra equipped with this product and  $\mathfrak{H}^0$  is its subalgebra. We denote these algebras by  $\mathfrak{H}_*^1$  and  $\mathfrak{H}_*^0$ . Hoffman showed a map  $Z : \mathfrak{H}_*^0 \rightarrow \mathbb{R}$  is homomorphic:

$$Z(w_1 * w_2) = Z(w_1)Z(w_2) \quad (w_1, w_2 \in \mathfrak{H}^0). \quad (2.4.1)$$

*Remark.* For example, the harmonic product of  $z_p z_q$  and  $z_r$  is given by

$$\begin{aligned} z_p * z_q z_r &= (1 * z_q z_r) z_p + (z_p * z_q) z_r + (1 * z_q) z_{p+r} \\ &= z_q z_r z_p + ((1 * z_q) z_p + (z_p * 1) z_q + (1 * 1) z_{p+q}) z_r + z_q z_{p+r} \\ &= z_q z_r z_p + z_q z_p z_r + z_p z_q z_r + z_{p+q} z_r + z_q z_{p+r}, \end{aligned}$$

which corresponds to

$$\zeta(p) \zeta(q, r) = \zeta(q, r, p) + \zeta(q, p, r) + \zeta(p, q, r) + \zeta(p + q, r) + \zeta(q, p + r).$$

Thus we often write (2.4.1) in terms of MZVs; for two indices  $\mathbf{k}$  and  $\mathbf{l}$ , we have

$$\zeta(\mathbf{k} * \mathbf{l}) = \zeta(\mathbf{k}) \zeta(\mathbf{l}).$$

In particular, as seen in the proof of Proposition 2.2.3, for a positive integer  $M$ , we have

$$\zeta_M(\mathbf{k} * \mathbf{l}) = \zeta_M(\mathbf{k}) \zeta_M(\mathbf{l}). \quad (2.4.2)$$

We also define the shuffle product.

**Definition 2.4.5.** The shuffle product  $\text{III}$  on  $\mathfrak{H}$  is defined by  $\mathbb{Q}$ -bilinearity and the following rules:

1.  $w \text{ III } 1 = 1 \text{ III } w = w$  for any word  $w \in \mathfrak{H}$ ,
2.  $(w_1 u_1 \text{ III } w_2 u_2) = (w_1 \text{ III } w_2 u_2) u_1 + (w_1 u_1 \text{ III } w_2) u_2$   
for any  $w_1, w_2 \in \mathfrak{H}$  and  $u_1, u_2 \in \{x, y\}$ .

In [24], Reutenauer proved the shuffle product is commutative and associative. This product gives  $\mathfrak{H}$  the structure of a  $\mathbb{Q}$ -algebra, which we denote by  $\mathfrak{H}_{\text{III}}$ . Obviously, the subspaces  $\mathfrak{H}^1$  and  $\mathfrak{H}^0$  become subalgebras of  $\mathfrak{H}$ , denoted by  $\mathfrak{H}_{\text{III}}^1$  and  $\mathfrak{H}_{\text{III}}^0$ , respectively. It is known that a map  $Z : \mathfrak{H}^0 \rightarrow \mathbb{R}$  is homomorphic under the shuffle product  $\text{III}$ :

$$Z(w_1 \text{ III } w_2) = Z(w_1)Z(w_2) \quad (w_1, w_2 \in \mathfrak{H}^0). \quad (2.4.3)$$

*Remark.* For example, the shuffle product of  $z_2 = yx$  and  $z_1 z_2 = yxy$  is given by

$$\begin{aligned} z_2 \text{ III } z_1 z_2 &= (y \text{ III } yyx)x + (yx \text{ III } yy)x \\ &= 3y y y x x + y y x y x + ((y \text{ III } yy)x + (yx \text{ III } y)y)x \\ &= 6y y y x x + 3y y x y x + y x y y x \\ &= 6z_1 z_1 z_3 + 3z_1 z_2 z_2 + z_2 z_1 z_2, \end{aligned}$$

which corresponds to

$$\zeta(2)\zeta(1, 2) = 6\zeta(1, 1, 3) + 3\zeta(1, 2, 2) + \zeta(2, 1, 2).$$

Thus we often write (2.4.3) in terms of MZVs; for two indices  $\mathbf{k}$  and  $\mathbf{l}$ , we have

$$\zeta(\mathbf{k} \text{ III } \mathbf{l}) = \zeta(\mathbf{k})\zeta(\mathbf{l}).$$

(2.4.1) and (2.4.3) lead to the finite double shuffle relations (FDS relations for short).

**Theorem 2.4.6 (FDS).** *For any elements  $w_1, w_2 \in \mathfrak{H}^0$ , we have*

$$Z(w_1 * w_2 - w_1 \text{ III } w_2) = 0.$$

Many relations among MZVs are obtained by the FDS relations. However, these relations are not enough for  $\dim_{\mathbb{Q}} \mathcal{Z}_k$  to satisfy the inequality in Theorem 2.2.2. For example, there exists 4 MZVs in weight 4, while the FDS relations of weight 4 are obviously only (2.3.7). So we need much more larger relations, which is the ‘‘regularization’’ procedure discussed in the next section.

## 2.5 Regularization of MZVs and extended double shuffle relations

In Theorem 2.4.6, we got non-trivial relations among MZVs by calculating the harmonic product and the shuffle product in  $\mathfrak{H}^0$ . We want to extend these multiplications to  $\mathfrak{H}^1$ . For this purpose, we introduce regularizations of MZVs.

**Proposition 2.5.1.** *We have two algebra homomorphisms*

$$Z^* : \mathfrak{H}_*^1 \rightarrow \mathbb{R}[T], \quad Z^{\text{III}} : \mathfrak{H}_{\text{III}}^1 \rightarrow \mathbb{R}[T]$$

that are uniquely characterized by the properties that they extend the evaluation map  $Z : \mathfrak{H} \rightarrow \mathbb{R}$  and send  $y$  to  $T$ .

*Proof.* In [12], by Hoffman's Theorem it is known that  $\mathfrak{H}_*^1 \simeq \mathfrak{H}_*^0[y]$ . Namely, any  $w \in \mathfrak{H}_*^1$  is uniquely expressed as

$$w = w_0 + y * w_1 + y^{*2} * w_2 + \cdots + y^{*n} * w_n \quad (w_0, w_1, \dots, w_n \in \mathfrak{H}^0),$$

where  $y^{*n} = \underbrace{y * \cdots * y}_n$ .  $Z^*(w)$  is obtained by replacing  $w_i$  with  $Z(w_i)$  and  $y$  with  $T$ , respectively.

As in the case of the shuffle product,  $\mathfrak{H}_{\text{III}}^1 \simeq \mathfrak{H}_{\text{III}}^0[y]$  is proved by Reutenauer in [24]. So for any  $w \in \mathfrak{H}_{\text{III}}^1$  is uniquely written as

$$w = w_0 + y \text{ III } w_1 + y^{\text{III}2} \text{ III } w_2 + \cdots + y^{\text{III}n} \text{ III } w_n \quad (w_0, w_1, \dots, w_n \in \mathfrak{H}^0),$$

where  $y^{\text{III}n} = \underbrace{y \text{ III } \cdots \text{ III } y}_n$ .  $Z^{\text{III}}(w)$  is given by replacing  $w_i$  with  $Z(w_i)$  and  $y$  with  $T$ . □

For an element  $w \in \mathfrak{H}^1$  expressed as above, we need its constant term to state a generalization of FDS later.

**Definition 2.5.2.** Let  $\bullet \in \{*, \text{III}\}$ . For  $w \in \mathfrak{H}^1$  with

$$w = w_0 + y \bullet w_1 + y^{\bullet 2} \bullet w_2 + \cdots + y^{\bullet n} \bullet w_n \quad (w_0, w_1, \dots, w_n \in \mathfrak{H}^0),$$

we define  $\text{reg}_{\bullet}(w) := w_0$ .

For an index  $\mathbf{k} = (k_1, \dots, k_r)$  (not necessarily admissible), the images of the corresponding word  $y x^{k_1-1} \cdots y x^{k_r-1}$  under the maps  $Z^*$  and  $Z^{\text{III}}$  are denoted by  $Z_{\mathbf{k}}^*(T)$  and  $Z_{\mathbf{k}}^{\text{III}}(T)$ , respectively. If  $\mathbf{k}$  is admissible, it is clear that  $Z_{\mathbf{k}}^*(T) = Z_{\mathbf{k}}^{\text{III}}(T) = \zeta(\mathbf{k})$ .

**Example 2.5.3.**

| $\mathbf{k}$    | $Z_{\mathbf{k}}^*(T)$   | $Z_{\mathbf{k}}^{\text{m}}(T)$                            |
|-----------------|---|---|
| (1)             | $T$   | $T$   |
| (1, 1)          | $\frac{T^2}{2} - \frac{\zeta(2)}{2}$  | $\frac{T^2}{2}$   |
| (2, 1)          | $\zeta(2)T - \zeta(3) - \zeta(1, 2)$  | $\zeta(2)T - 2\zeta(1, 2)$                                |
| (1, 1, 1)       | $\frac{T^3}{6} - \frac{\zeta(2)}{2}T + \frac{\zeta(3)}{3}$  | $\frac{T^3}{6}$   |
| (3, 1)          | $\zeta(3)T - \zeta(4) - \zeta(3, 1)$  | $\zeta(3)T - 2\zeta(1, 3) - \zeta(2, 2)$                  |
| (1, 2, 1)       | $\zeta(2, 1)T - \zeta(1, 3) - \zeta(2, 2) - 2\zeta(1, 1, 2)$  | $\zeta(1, 2)T - 3\zeta(1, 1, 2)$                          |
| (2, 1, 1)       | $\frac{\zeta(2)}{2}T^2 - (\zeta(3) + \zeta(1, 2))T$<br>$+ \frac{\zeta(4)}{2} + \zeta(1, 3) + \zeta(1, 1, 2)$  | $\frac{\zeta(2)}{2}T^2 - 2\zeta(1, 2)T + 3\zeta(1, 1, 2)$ |
| (1, 1, 1, 1)    | $\frac{T^4}{24} - \frac{\zeta(2)}{4}T^2 + \frac{\zeta(3)}{3}T - \frac{\zeta(4)}{8} + \frac{\zeta(2, 2)}{4}$   | $\frac{T^4}{24}$  |
| (1, 1, 1, 1, 1) | $\frac{T^5}{120} - \frac{\zeta(2)}{12}T^3 + \frac{\zeta(3)}{6}T^2$<br>$- \left( \frac{\zeta(4)}{8} - \frac{\zeta(2, 2)}{4} \right)T + \frac{\zeta(5)}{30}$<br>$- \frac{\zeta(3, 2)}{6} - \frac{\zeta(2, 3)}{6}$ | $\frac{T^5}{120}$   |

A fundamental theorem of regularizations of MZVs says that two polynomials  $Z_{\mathbf{k}}^*(T)$  and  $Z_{\mathbf{k}}^{\text{m}}(T)$  are related to each other by a simple  $\mathbb{R}$ -linear map  $\rho$  described in terms of the Taylor series of the Gamma function  $\Gamma(z)$ .

**Definition 2.5.4.** We define an  $\mathbb{R}$ -linear map  $\rho : \mathbb{R}[T] \rightarrow \mathbb{R}[T]$  as

$$\rho(e^{Tu}) = A(u)e^{Tu}, \quad (2.5.1)$$

where

$$A(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n)u^n\right) (= \Gamma(1+u)e^{\gamma u}),$$

with Euler's constant defined as

$$\gamma := \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right). \quad (2.5.2)$$

Note that  $\rho$  on the left-hand side of (2.5.1) operates coefficientwise in  $u$ . Moreover,  $\rho$  is an invertible  $\mathbb{R}$ -linear map, and the inverse map of  $\rho$  is given by

$$\rho^{-1}(e^{Tu}) = A(u)^{-1}e^{Tu}.$$

**Example 2.5.5.** Since

$$\begin{aligned} A(u) &= 1 + \left( \frac{\zeta(2)}{2}u^2 - \frac{\zeta(3)}{3}u^3 + \frac{\zeta(4)}{4}u^4 - \cdots \right) + \frac{1}{2} \left( \frac{\zeta(2)}{2}u^2 - \frac{\zeta(3)}{3}u^3 + \frac{\zeta(4)}{4}u^4 - \cdots \right)^2 \\ &= 1 + \frac{\zeta(2)}{2}u^2 - \frac{\zeta(3)}{3}u^3 + \left( \frac{\zeta(4)}{4} + \frac{\zeta(2)^2}{8} \right)u^4 + \cdots, \end{aligned}$$

we have

$$\begin{aligned} \rho(e^{Tu}) &= \rho(1) + \rho(T)u + \rho(T^2)\frac{u^2}{2} + \rho(T^3)\frac{u^3}{6} + \rho(T^4)\frac{u^4}{24} + \cdots, \\ A(u)e^{Tu} &= \left( 1 + \frac{\zeta(2)}{2}u^2 - \frac{\zeta(3)}{3}u^3 + \left( \frac{\zeta(4)}{4} + \frac{\zeta(2)^2}{8} \right)u^4 + \cdots \right) \\ &\quad \times \left( 1 + Tu + T^2\frac{u^2}{2} + T^3\frac{u^3}{6} + T^4\frac{u^4}{24} + \cdots \right) \\ &= 1 + Tu + (T^2 + \zeta(2))\frac{u^2}{2} + (T^3 + 3\zeta(2)T - 2\zeta(3))\frac{u^3}{6} \\ &\quad + (T^4 + 6\zeta(2)T^2 - 8\zeta(3)T + 6\zeta(4) + 3\zeta(2)^2)\frac{u^4}{24} + \cdots, \end{aligned}$$

and hence

$$\begin{aligned} \rho(1) &= 1, \\ \rho(T) &= T, \\ \rho(T^2) &= T^2 + \zeta(2), \\ \rho(T^3) &= T^3 + 3\zeta(2)T - 2\zeta(3), \\ \rho(T^4) &= T^4 + 6\zeta(2)T^2 - 8\zeta(3)T + 6\zeta(4) + 3\zeta(2)^2. \end{aligned}$$

Between two regularized maps in Proposition 2.5.1, the following relation holds.

**Theorem 2.5.6** (Zagier (cf. [13])). *On  $\mathfrak{H}^1$ , we have*

$$Z^{\text{III}} = \rho \circ Z^*.$$

Theorem 2.5.6 leads to a “generalization” of FDS relations.

**Theorem 2.5.7** (Extended double shuffle relations (EDS relations), Ihara–Kaneko–Zagier [13]). *For  $w_1 \in \mathfrak{H}^1$  and  $w_0 \in \mathfrak{H}^0$ , we have*

$$Z^{\text{III}}(w_1 \text{ III } w_0 - w_1 * w_0) = 0, \quad Z^*(w_1 \text{ III } w_0 - w_1 * w_0) = 0.$$

*In particular,*

$$Z(\text{reg}_{\text{III}}(w_1 \text{ III } w_0 - w_1 * w_0)) = 0, \quad Z(\text{reg}_*(w_1 \text{ III } w_0 - w_1 * w_0)) = 0$$

*holds.*

The authors of [13] conjectured the following.

**Conjecture 2.5.8** (Ihara–Kaneko–Zagier [13]). *The EDS relations exhaust all relations among MZVs.*

## 2.6 Linear relations of MZVs

From Theorem 2.2.2, we see that there are many identities among MZVs. In this section, we state well-known values and  $\mathbb{Q}$ -linear relations about MZVs.

The special value of the Riemann zeta function

$$\zeta(2) = \frac{\pi^2}{6} \tag{2.6.1}$$

is famous. We give generalizations of  $\zeta(2)$  from the viewpoint of MZVs.

**Proposition 2.6.1.** *For  $n \in \mathbb{Z}_{\geq 1}$ , we have*

$$\zeta(\underbrace{2, \dots, 2}_n) = \frac{\pi^{2n}}{(2n+1)!}. \tag{2.6.2}$$

*Proof.* By expanding the infinite product of  $\sin \pi x / \pi x$ , we have

$$\begin{aligned}
\frac{\sin \pi x}{\pi x} &= \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2}\right) = \left(1 - \frac{x^2}{1^2}\right) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right) \times \dots \\
&= 1 - \left(\sum_{m=1}^{\infty} \frac{1}{m^2}\right) x^2 + \left(\sum_{0 < m_1 < m_2} \frac{1}{m_1^2 m_2^2}\right) x^4 \\
&\quad - \left(\sum_{0 < m_1 < m_2 < m_3} \frac{1}{m_1^2 m_2^2 m_3^2}\right) x^6 + \dots \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\underbrace{2, \dots, 2}_n) x^{2n}.
\end{aligned} \tag{2.6.3}$$

On the other hand, by the Taylor expansion, we have

$$\frac{\sin \pi x}{\pi x} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n+1)!} x^{2n}. \tag{2.6.4}$$

Comparing the coefficient of  $x^{2n}$  of (2.6.3) and (2.6.4), we have the assertion.  $\square$

*Remark.* It is clear that if  $n = 1$ , then (2.6.2) reduces to (2.6.1).

It is also known that  $\zeta(3)$  is an irrational number. We do not even know the transcendence of  $\zeta(3)$ . However, we have an expression of  $\zeta(3)$  in terms of MZVs.

**Proposition 2.6.2.**

$$\zeta(1, 2) = \zeta(3). \tag{2.6.5}$$

*Proof.* By the sum

$$\zeta(1, 2) + \zeta(3) = \sum_{0 < m \leq n} \frac{1}{m n^2} = \sum_{n=1}^{\infty} \left(\sum_{m=1}^n \frac{1}{m}\right) \frac{1}{n^2},$$

and the telescoping sum

$$\sum_{m=1}^n \frac{1}{m} = \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+n}\right),$$

we have

$$\zeta(1, 2) + \zeta(3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+n}\right) \frac{1}{n^2}.$$

Since

$$\left(\frac{1}{m} - \frac{1}{m+n}\right) \frac{1}{n^2} = \frac{1}{mn(m+n)} = \left(\frac{1}{m} + \frac{1}{n}\right) \frac{1}{(m+n)^2},$$

we obtain

$$\zeta(1, 2) + \zeta(3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{m(m+n)^2} + \frac{1}{n(m+n)^2} \right) = 2\zeta(1, 2),$$

which leads to the equality.  $\square$

Note that (2.6.5) is a special case of the sum formula which is of a class of linear relations among MZVs.

Next, we state several relations about MZVs without proof. Let

$$\begin{aligned} I(k, r, s) &:= \{\mathbf{k} : \text{index} \mid \text{wt}(\mathbf{k}) = k, \text{dep}(\mathbf{k}) = r, \text{ht}(\mathbf{k}) = s\}, \\ I_0(k, r, s) &:= \{\mathbf{k} : \text{admissible} \mid \text{wt}(\mathbf{k}) = k, \text{dep}(\mathbf{k}) = r, \text{ht}(\mathbf{k}) = s\}. \end{aligned}$$

Furthermore, let  $I_0(k, r, *) = \bigcup_s I_0(k, r, s)$  and  $I_0(k, *, s) = \bigcup_r I_0(k, r, s)$ . Then we have  $I(k, r, s) \neq \emptyset$  if and only if  $(s = 0 \wedge k = r)$  or  $(r \geq s \geq 1 \wedge k \geq r + s)$ .

**Theorem 2.6.3** (Sum formula, Granville [8], Zagier [28], Hoffman [10]). *For  $k, r \in \mathbb{Z}_{>0}$  with  $k > r$ , we have*

$$\sum_{\mathbf{k} \in I_0(k, r, *)} \zeta(\mathbf{k}) = \zeta(k), \quad (2.6.6)$$

$$\sum_{\mathbf{k} \in I_0(k, r, *)} \zeta^*(\mathbf{k}) = \binom{k-1}{r-1} \zeta(k). \quad (2.6.7)$$

The first equation is conjectured by Moen and Hoffman, and proved by Granville and Zagier, independently.

The sum in (2.6.6) runs over all the indices of a fixed weight and depth, and so does that in (2.6.7). Similarly, by taking the sum over all the indices of a fixed weight and height, we have the following relations.

**Theorem 2.6.4** (Le–Murakami’s relation, Le–Murakami [18]). *For  $k, s \in \mathbb{Z}_{>0}$  with  $k \geq s$ , we have*

$$\sum_{\mathbf{k} \in I_0(2k, *, s)} (-1)^{\text{dep}(\mathbf{k})} \zeta(\mathbf{k}) = \frac{(-1)^k}{(2k+1)!} \sum_{r=0}^{k-s} \binom{2k+1}{2r} (2-2^r) B_{2r} \pi^{2k},$$

where  $B_n$  is the  $n$ -th Seki–Bernoulli number defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{te^t}{e^t - 1}.$$

We remark that the sum with  $2k$  replaced by  $2k + 1$  satisfies

$$\sum_{\mathbf{k} \in I_0(2k+1, *, s)} (-1)^{\text{dep}(\mathbf{k})} \zeta(\mathbf{k}) = 0$$

holds for  $k \geq s \geq 1$  by Theorem 2.3.8.

**Theorem 2.6.5** (Aoki–Ohno’s relation, Aoki–Ohno[1]). *For  $k, s \in \mathbb{Z}_{>0}$  with  $k \geq 2s$ , we have*

$$\sum_{\mathbf{k} \in I_0(k, *, s)} \zeta^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \zeta(k). \quad (2.6.8)$$

In [2], for integers  $k, r, s$  with  $k \geq r + s$  and  $r \geq s \geq 1$ , Aoki, Kombu and Ohno introduced the generating function  $\Phi_0^*(x, y, z)$  of the sum of all MZVs whose weight, depth and height are given by  $k, r$  and  $s$ , respectively:

$$\Phi_0^*(x, y, z) := \sum_{k, r, s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k, r, s)} \zeta^*(\mathbf{k}) \right) x^{k-r-s} y^{r-s} z^{2s-2}.$$

Furthermore, by using the generalized hypergeometric function defined as (see [4])

$${}_pF_q \left( \begin{matrix} A_1, \dots, A_p \\ B_1, \dots, B_q \end{matrix} ; t \right) = \sum_{n=0}^{\infty} \frac{(A_1)_n \cdots (A_p)_n}{n! (B_1)_n \cdots (B_q)_n} t^n, \quad (2.6.9)$$

for  $p, q \in \mathbb{Z}_{\geq 0}$  with the Pochhammer symbol  $(x)_n$  given by

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x+1)(x+2) \cdots (x+n-1) & (n > 0), \end{cases}$$

the authors of [2] proved the following Theorem.

**Theorem 2.6.6** (Aoki–Kombu–Ohno [2]). *For integers  $k, r, s$ , we have*

$$\Phi_0^*(x, y, z) = \frac{1}{(1-\alpha)(1-\beta)} {}_3F_2 \left( \begin{matrix} 1-x, 1, 1 \\ 2-\alpha, 2-\beta \end{matrix} ; 1 \right), \quad (2.6.10)$$

where  $\alpha$  and  $\beta$  are indeterminates with

$$\alpha + \beta = x + y, \quad \alpha\beta = xy - z^2.$$

It is interesting that by specializing  $xy = z^2$  in (2.6.10), we obtain the sum formula (2.6.7) and specializing  $x = y$  in (2.6.10), we get Aoki–Ohno’s relation (2.6.8). In [19], Li proved the following result with this generating function  $\Phi_0^*(x, y, z)$ .

**Theorem 2.6.7** (Li, [19]). *We have*

$$x\Phi_0^*(-x, y, z) - y\Phi_0^*(-y, x, z) = Z(x, y), \quad (2.6.11)$$

where

$$Z(x, y) = \frac{x-y}{ab} + A(x, y) \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(x+a)\Gamma(x+b)}{\Gamma(x)\Gamma(y)}$$

with

$$\begin{aligned} A(x, y) &= \frac{1}{\Gamma(x+a)\Gamma(1-(x+a))} \left( \frac{\Gamma(y)\Gamma(1-y)}{\Gamma(a)\Gamma(1-a)} + \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(b)\Gamma(1-b)} \right) \\ &= \frac{(\cos \pi(x+y) - \cos \pi \sqrt{(x+y)^2 + 4z^2}) \sin \pi(x-y)}{2\pi \sin \pi x \sin \pi y}, \end{aligned}$$

and

$$a+b = -x+y, \quad ab = -xy - z^2. \quad (2.6.12)$$

Moreover, by using the well-known identity for the Gamma function

$$\Gamma(1+x) = \exp\left(\gamma x + \sum_{m=2}^{\infty} \frac{\zeta(m)}{m} x^m\right) \quad (|x| < 1), \quad (2.6.13)$$

Li showed that Theorem 2.6.7 implies the following corollary which was conjectured by Kaneko–Ohno in [15].

**Corollary 2.6.8** (Li [19], Kaneko–Ohno [15]). *For any positive integers  $m, n, s$  with  $m, n \geq s$ , we have*

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta^*(\mathbf{k}) - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta^*(\mathbf{k}) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \zeta(7), \dots].$$

# Chapter 3

## Finite multiple zeta values and Symmetric multiple zeta values

In this chapter, we state two finite analogues of MZVs. First, we define the finite multiple zeta(-star) values (FMZ(S)V) for short) as elements of a  $\mathbb{Q}$ -algebra  $\mathcal{A}$ . Secondly, we define the symmetric multiple zeta(-star) values (SMZ(S)V) for short) which are elements of  $\mathcal{Z}/\zeta(2)\mathcal{Z}$  as real counterparts of FMZ(S)V. Lastly, we state the Kaneko–Zagier conjecture that has a certain correspondence between FMZVs and SMZVs and introduce several relations for FMZVs and SMZVs.

### 3.1 Notation for FMZVs

In [17], Kaneko and Zagier introduced the ring

$$\mathcal{A} := \left( \prod_{p:\text{prime}} \mathbb{Z}/p\mathbb{Z} \right) / \left( \bigoplus_{p:\text{prime}} \mathbb{Z}/p\mathbb{Z} \right) = \{(a_p)_p \mid a_p \in \mathbb{Z}/p\mathbb{Z}\} / \sim, \quad (3.1.1)$$

where  $p$  runs over all prime numbers and the relation  $(a_p)_p \sim (b_p)_p$  means that the equality  $a_p = b_p$  holds in  $\mathbb{Z}/p\mathbb{Z}$  for all but finitely many primes  $p$ . Componentwise addition and multiplication equip  $\mathcal{A}$  with the structure of a ring. Moreover, for  $r \in \mathbb{Q}$ , by putting  $r_p$  as

$$r_p := \begin{cases} r \bmod p & \text{if } \gcd(\text{denominator of } r, p) = 1, \\ 0 & \text{if } \gcd(\text{denominator of } r, p) \neq 1, \end{cases}$$

an injective map  $\mathbb{Q} \ni r \mapsto (r_p)_p \in \mathcal{A}$  gives a  $\mathbb{Q}$ -algebra structure.

For an index  $\mathbf{k} = (k_1, \dots, k_r)$  and a positive integer  $M$ , we set

$$\zeta_M^*(\mathbf{k}) = \zeta_M^*(k_1, \dots, k_r) := \sum_{0 < m_1 \leq \dots \leq m_r \leq M} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

**Definition 3.1.1** (FMZV and FMZSV). For an index  $\mathbf{k} = (k_1, \dots, k_r)$ , FMZV and FMZSV are defined by

$$\begin{aligned} \zeta_{\mathcal{A}}(\mathbf{k}) &= \zeta_{\mathcal{A}}(k_1, \dots, k_r) := (\zeta_{p-1}(k_1, \dots, k_r) \bmod p)_p \in \mathcal{A}, \\ \zeta_{\mathcal{A}}^*(\mathbf{k}) &= \zeta_{\mathcal{A}}^*(k_1, \dots, k_r) := (\zeta_{p-1}^*(k_1, \dots, k_r) \bmod p)_p \in \mathcal{A}. \end{aligned}$$

Similarly as in the case of ordinary real MZVs, we obtain the following proposition with the same proof as for Proposition 2.1.2.

**Proposition 3.1.2.** For an index  $\mathbf{k}$ , we have,

$$\zeta_{\mathcal{A}}^*(\mathbf{k}) = \sum_{\mathbf{k}' \leq \mathbf{k}} \zeta_{\mathcal{A}}(\mathbf{k}').$$

*Remark.* Although some components of an index  $\mathbf{k}$  can be non-positive integers because  $\zeta_p(\mathbf{k})$  is a finite sum, we restrict ourselves to positive integer components. In fact, by using Seki–Bernoulli formula for sum of powers, we easily find that if an index  $\mathbf{k}$  includes non-positive integer components, then  $\zeta_{\mathcal{A}}(\mathbf{k})$  is expressed as a linear combination of certain FMZVs whose indices have positive integer components: for example, we have

$$\begin{aligned} \zeta_{\mathcal{A}}(-1, 3, 2) &= \left( \sum_{0 < \ell < m < n \leq p} \frac{1}{\ell^{-1} m^3 n^2} \bmod p \right)_p \\ &= \left( \sum_{0 < m < n \leq p} \left( \sum_{\ell=1}^{m-1} \ell \right) \frac{1}{m^3 n^2} \bmod p \right)_p \\ &= \frac{1}{2} \left( \sum_{0 < m < n \leq p} \frac{m^2 - m}{m^3 n^2} \bmod p \right)_p \\ &= \frac{1}{2} \left( \sum_{0 < m < n \leq p} \frac{1}{m n^2} \bmod p \right)_p - \frac{1}{2} \left( \sum_{0 < m < n \leq p} \frac{1}{m^2 n^2} \bmod p \right)_p \\ &= \frac{1}{2} \zeta_{\mathcal{A}}(1, 2) - \frac{1}{2} \zeta_{\mathcal{A}}(2, 2). \end{aligned}$$

Next, we define the  $\mathbb{Q}$ -vector spaces which are spanned by FMZVs.

**Definition 3.1.3.** For a non-negative integer  $k$ , the  $\mathbb{Q}$ -vector spaces  $\{\mathcal{Z}_{\mathcal{A},k}\}$  are defined by

$$\begin{aligned}\mathcal{Z}_{\mathcal{A},0} &:= \mathbb{Q}, \\ \mathcal{Z}_{\mathcal{A},k} &= \sum_{\substack{\mathbf{k}: \text{admissible} \\ \text{wt}(\mathbf{k})=k}} \mathbb{Q} \cdot \zeta_{\mathcal{A}}(\mathbf{k}) \quad (k \geq 1),\end{aligned}$$

and  $\mathcal{Z}_{\mathcal{A}}$  is defined by

$$\mathcal{Z}_{\mathcal{A}} := \sum_{k=0}^{\infty} \mathcal{Z}_{\mathcal{A},k}.$$

Zagier conjectured the following identity concerning the dimension of  $\mathcal{Z}_{\mathcal{A},k}$ , for all non-negative integers  $k$ :

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} = d_{k-3},$$

where the sequence  $d_k$  is given by

$$d_{-2} := 0, \quad d_{-1} := 0, \quad d_0 := 1, \quad d_k := d_{k-2} + d_{k-3} \quad (k \geq 1).$$

Recently, Yasuda announced that  $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} \leq d_{k-3}$  holds for all  $k \in \mathbb{Z}_{\geq 0}$  by using Akagi–Hirose–Yasuda’s results and Jarrosay’s results. So, we find that there exist many  $\mathbb{Q}$ -linear relations among FMZVs and it is also important to study relations among FMZVs as in the cases of ordinary MZVs.

For two indices  $\mathbf{k}, \mathbf{l}$ , we understand  $\zeta_{\mathcal{A}}(\mathbf{k} * \mathbf{l})$  as the expansion of  $\zeta(\mathbf{k} * \mathbf{l})$  with formally replacing “ $\zeta$ ” with “ $\zeta_{\mathcal{A}}$ ”. Then we have the following theorem.

**Theorem 3.1.4** (Kanko–Zagier [17]). *For any two indices  $\mathbf{k}, \mathbf{l}$ , we have*

$$\zeta_{\mathcal{A}}(\mathbf{k} * \mathbf{l}) = \zeta_{\mathcal{A}}(\mathbf{k})\zeta_{\mathcal{A}}(\mathbf{l}). \quad (3.1.2)$$

Theorem 3.1.4 follows from the fact that the finite truncated sum  $\zeta_p(\mathbf{k})$  satisfies the harmonic rule (2.4.2). Thus (3.1.2) equip  $\mathcal{Z}_{\mathcal{A}}$  with a structure of  $\mathbb{Q}$ -algebra. As for the shuffle product, we do not have any formula involving products. However, by understanding  $\zeta_{\mathcal{A}}(\mathbf{k} \amalg \mathbf{l})$  as the expansion of  $\zeta(\mathbf{k} \amalg \mathbf{l})$  with formally replacing “ $\zeta$ ” with “ $\zeta_{\mathcal{A}}$ ” for two indices  $\mathbf{k}$  and  $\mathbf{l}$ , we have the following relation.

**Theorem 3.1.5** (Kaneko–Zagier [17]). *For indices  $\mathbf{k}$  and  $\mathbf{l}$ , we have*

$$\zeta_{\mathcal{A}}(\mathbf{k} \amalg \mathbf{l}) = (-1)^{\text{wt}(\mathbf{l})} \zeta_{\mathcal{A}}(\mathbf{k}, \bar{\mathbf{l}}), \quad (3.1.3)$$

where  $\bar{\mathbf{l}} = (l_s, \dots, l_1)$  for  $\mathbf{l} = (l_1, \dots, l_s)$ .

Note that any relations in  $\mathcal{Z}_{\mathcal{A}}$  for FMZVs are to be deduced from (3.1.2) and (3.1.3), which is conjectured by Kaneko–Zagier in [17].

To close this section, we state a short comment about the difference between MZVs and FMZVs. By definition of FMZVs, we have

$$\begin{aligned}\zeta_{\mathcal{A}}(2) &= \left( \sum_{0 < m < p} \frac{1}{m^2} \bmod p \right)_p \\ &= (1, 2, 0, 0, 0, \dots) \\ &= (0)_p.\end{aligned}\tag{3.1.4}$$

However, as observed in (2.6.5),  $\zeta(2) = \pi^2/6$ . So  $\zeta(\mathbf{k}) \in \mathbb{R}$  is not regarded as a good counterpart of  $\zeta_{\mathcal{A}}(\mathbf{k}) \in \mathcal{A}$  (see Figure 3.1). In the next section, we introduce a “good” counterpart for FMZVs.

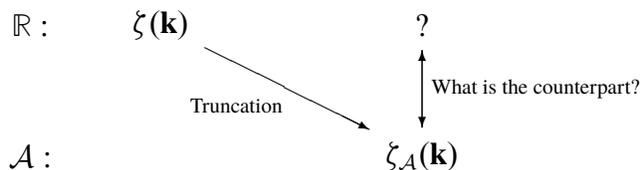


Figure 3.1: MZVs and FMZVs

## 3.2 Notation for SMZVs and Kaneko–Zagier conjecture

In this section, we define SMZVs, which are counterparts of FMZVs in  $\mathbb{R}$ . According to [17], for an index  $\mathbf{k} = (k_1, \dots, k_r)$  we set

$$\zeta_{\mathcal{S}}^*(\mathbf{k}) = \zeta_{\mathcal{S}}^*(k_1, \dots, k_r) := \sum_{i=0}^r (-1)^{k_r + \dots + k_{i+1}} \zeta^*(k_1, \dots, k_i; T) \zeta^*(k_r, \dots, k_{i+1}; T), \tag{3.2.1}$$

$$\zeta_{\mathcal{S}}^{\text{III}}(\mathbf{k}) = \zeta_{\mathcal{S}}^{\text{III}}(k_1, \dots, k_r) := \sum_{i=0}^r (-1)^{k_r + \dots + k_{i+1}} \zeta^{\text{III}}(k_1, \dots, k_i; T) \zeta^{\text{III}}(k_r, \dots, k_{i+1}; T), \tag{3.2.2}$$

and for indices  $\mathbf{k}$  and  $\mathbf{k}'$ , put

$$\begin{aligned}\zeta_{\mathcal{S}}^{*,*}(\mathbf{k}) &= \sum_{\mathbf{k} \leq \mathbf{k}'} \zeta_{\mathcal{S}}^*(\mathbf{k}'), \\ \zeta_{\mathcal{S}}^{*,\text{III}}(\mathbf{k}) &= \sum_{\mathbf{k} \leq \mathbf{k}'} \zeta_{\mathcal{S}}^{\text{III}}(\mathbf{k}'),\end{aligned}$$

where  $\zeta^*(\mathbf{k}; T)$  and  $\zeta^{\text{III}}(\mathbf{k}; T)$  are the harmonic regularized polynomial and the shuffle regularized polynomial introduced in section 2.5, respectively:

$$\begin{aligned}\zeta^*(\mathbf{k}; T) &= Z_{\mathbf{k}}^*(T) \in \mathbb{R}[T], \\ \zeta^{\text{III}}(\mathbf{k}; T) &= Z_{\mathbf{k}}^{\text{III}}(T) \in \mathbb{R}[T].\end{aligned}$$

Although (3.2.1) and (3.2.2) seem to be polynomials in  $T$ ,  $\zeta_S^*(\mathbf{k}; T)$  and  $\zeta_S^{\text{III}}(\mathbf{k}; T)$  are indeed real numbers independent of  $T$  despite their appearance.

**Proposition 3.2.1** (Kaneko–Zagier [17]). *For any index  $\mathbf{k}$ , we have*

$$\zeta_S^*(\mathbf{k}), \zeta_S^{\text{III}}(\mathbf{k}) \in \mathbb{R}.$$

In particular, we have

$$\begin{aligned}\zeta_S^*(\mathbf{k}) &= \sum_{i=0}^r (-1)^{k_r + \dots + k_{i+1}} \zeta^*(k_1, \dots, k_i; 0) \zeta^*(k_r, \dots, k_{i+1}; 0), \\ \zeta_S^{\text{III}}(\mathbf{k}) &= \sum_{i=0}^r (-1)^{k_r + \dots + k_{i+1}} \zeta^{\text{III}}(k_1, \dots, k_i; 0) \zeta^{\text{III}}(k_r, \dots, k_{i+1}; 0).\end{aligned}$$

As mentioned in the first chapter, two types regularized polynomials  $\zeta^*(\mathbf{k}; T)$  and  $\zeta^{\text{III}}(\mathbf{k}; T)$  satisfy  $\zeta^{\text{III}}(\mathbf{k}; T) = \rho(\zeta^*(\mathbf{k}; T))$  by the  $\mathbb{Q}$ -linear map  $\rho$ . As for the difference of  $\zeta_S^*(\mathbf{k})$  and  $\zeta_S^{\text{III}}(\mathbf{k}; T)$ , the following result holds.

**Theorem 3.2.2** (Kaneko–Zagier (cf. [17])). *For any index  $\mathbf{k}$ , we have*

$$\zeta_S^*(\mathbf{k}) - \zeta_S^{\text{III}}(\mathbf{k}) \in \zeta(2)\mathcal{Z}.$$

By Theorem 3.2.2 and the relation “ $\preceq$ ” for indices, we define SMZVs and SMZSVs as follows.

**Definition 3.2.3** (SMZV and SMZSV). For indices  $\mathbf{k}$  and  $\mathbf{k}'$ , SMZV and SMZSV are defined as

$$\begin{aligned}\zeta_S(\mathbf{k}) &:= \zeta_S^\bullet(\mathbf{k}) \bmod \zeta(2)\mathcal{Z}, \\ \zeta_S^*(\mathbf{k}) &:= \sum_{\mathbf{k} \preceq \mathbf{k}'} \zeta_S(\mathbf{k}'),\end{aligned}$$

where  $\bullet \in \{*, \text{III}\}$ .

By the conditions above, we have

$$\begin{aligned}\zeta_S(2) &= (-1)^2 \zeta(2) + \zeta(2) \\ &= 2\zeta(2) \\ &\equiv 0 \pmod{\zeta(2)\mathcal{Z}}.\end{aligned}$$

Since we observed that  $\zeta_{\mathcal{A}}(2) = (0)_p$  holds in (3.1.4), it seems that  $\zeta_S(\mathbf{k})$  is a good counterpart in  $\mathbb{R}$  for  $\zeta_{\mathcal{A}}(\mathbf{k})$ .

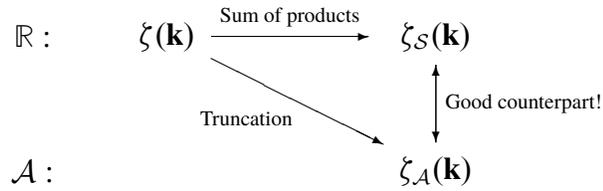


Figure 3.2: FMZVs and SMZVs

This is not a coincidence. Kaneko and Zagier suggested the following surprising correspondence, which is said to be the Kaneko–Zagier conjecture.

**Conjecture 3.2.4** (Kaneko–Zagier (cf. [17])). *There exists an isomorphism as a  $\mathbb{Q}$ -algebra such that*

$$\phi_{KZ} : \mathcal{Z}_{\mathcal{A}} \rightarrow \mathcal{Z}/\zeta(2)\mathcal{Z}; \quad \zeta_{\mathcal{A}}(\mathbf{k}) \mapsto \zeta_S(\mathbf{k}).$$

The Kaneko–Zagier conjecture means that any relations among  $\zeta_{\mathcal{A}}(\mathbf{k})$  should hold in exactly the same form in  $\mathcal{Z}/\zeta(2)\mathcal{Z}$  as  $\zeta_S(\mathbf{k})$  and vice versa. This conjecture is far from being solved at the present time. However, many relations for  $\zeta_{\mathcal{A}}(\mathbf{k})$  and  $\zeta_S(\mathbf{k})$  are shown to take the same form, which provide partial evidence for this conjecture.

In the next section, we state several values and relations which hold for FMZVs and SMZVs.

### 3.3 Linear relations of FMZVs and SMZVs

Here and hereafter, to describe properties of FMZVs and SMZVs, we put  $\mathcal{F} \in \{\mathcal{A}, \mathcal{S}\}$  and for a positive integer  $k$ , set

$$\mathfrak{z}_{\mathcal{F}}(k) := \begin{cases} \left( \binom{B_{p-k}}{k} \pmod{p} \right)_p & (\mathcal{F} = \mathcal{A}), \\ \zeta(k) \pmod{\zeta(2)\mathcal{Z}} & (\mathcal{F} = \mathcal{S}), \end{cases}$$

where  $B_k$  is the Seki–Bernoulli number. We remark that for finitely many prime numbers  $p$  with  $p - k < 0$ , the Seki–Bernoulli numbers  $B_{p-k}$  are not defined. However, we only consider the large enough prime numbers  $p$  because finite number of components in  $\mathcal{A}$  are ignored (see (3.1.1)).

As in the cases  $\zeta_{\mathcal{A}}(2) = (0)_p$  and  $\zeta_{\mathcal{S}}(2) = 0$ , it is known that all values  $\zeta_{\mathcal{F}}(\mathbf{k})$  of single index vanish.

**Theorem 3.3.1** (Hoffman [11]). *For  $k \in \mathbb{Z}_{>0}$ , we have*

$$\zeta_{\mathcal{F}}(k) = \zeta_{\mathcal{F}}^*(k) = 0.$$

*Proof.* It is clear that  $\zeta_{\mathcal{F}}(k) = \zeta_{\mathcal{F}}^*(k)$  by definition.

i)  $\mathcal{F} = \mathcal{A}$ .

Since  $\zeta_p(k) = 0 \pmod{p}$  holds when  $p - 1 \nmid k$  by Fermat's little theorem, the assertion follows.

ii)  $\mathcal{F} = \mathcal{S}$ .

By definition of SMZVs,

$$\begin{aligned} \zeta_{\mathcal{S}}^*(k) &= (-1)^k \zeta^*(k) + \zeta^*(k) \\ &= \begin{cases} 0 & (k : \text{odd}), \\ 2\zeta(k) & (k : \text{even}), \end{cases} \end{aligned}$$

then we have  $\zeta_{\mathcal{S}}^*(k) \equiv 0 \pmod{\zeta(2)\mathcal{Z}}$ .

This completes the proof. □

Similarly, the cases of  $\text{dep}(\mathbf{k}) = 2$  and  $\text{dep}(\mathbf{k}) = 3$  are given as follows.

**Theorem 3.3.2** (Hoffman [11]). *For  $k_1, k_2 \in \mathbb{Z}_{>0}$ , we have*

$$\zeta_{\mathcal{F}}(k_1, k_2) = \zeta_{\mathcal{F}}^*(k_1, k_2) = (-1)^{k_2} \binom{k_1 + k_2}{k_1} \mathfrak{Z}_{\mathcal{F}}(k_1 + k_2).$$

**Theorem 3.3.3** (Hoffman [11]). *For an index  $(k_1, k_2, k_3)$  of odd weight, we have*

$$\begin{aligned} -\zeta_{\mathcal{F}}(k_1, k_2, k_3) &= \zeta_{\mathcal{F}}^*(k_1, k_2, k_3) \\ &= \frac{1}{2} \left( (-1)^{k_1} \binom{k_1 + k_2 + k_3}{k_1} - (-1)^{k_3} \binom{k_1 + k_2 + k_3}{k_3} \right) \mathfrak{Z}_{\mathcal{F}}(k_1 + k_2 + k_3). \end{aligned}$$

In this section, we state several relations for FMZVs and SMZVs. As seen in (3.1.2), FMZVs satisfy the harmonic relation because each  $p$  component  $\zeta_{p-1}(\mathbf{k})$  satisfies the harmonic rule. According to the Kaneko–Zagier conjecture, SMZVs should also satisfy the same rule. Although this fact is checked directly by the definition of SMZVs. In this thesis, we introduce Kontsevich’s suggestion. He proposed that “ $\sum_{0 < m_1 < m_2 < 0}$ ” realizes the definition of  $\zeta_S^*(\mathbf{k})$  as the sum  $\sum_{0 < m_1 < m_2 < p}$  is indeed the definition of  $\zeta_{\mathcal{A}}(k_1, k_2)$ . This is mathematically formulated as follows; by setting the order

$$1 < 2 < \cdots < (\infty = -\infty) < \cdots < -2 < -1, \quad (3.3.1)$$

we have  $\sum_{m_1 < m_2} 1/m_1^{k_1} m_2^{k_2} = \zeta_S^*(k_1, k_2)$  (see Proposition 3.3.4 below). This order defined for non-zero integers is called Kontsevich’s order. We remark that the symbol “ $<$ ” for  $\mathbb{Z} \setminus \{0\}$  should not be confused with the same one for indices. See Figure 3.3 and Figure 3.3. Comparing the modulo  $p$  order and Kontsevich’s order, we see that these two worlds are “similar”.

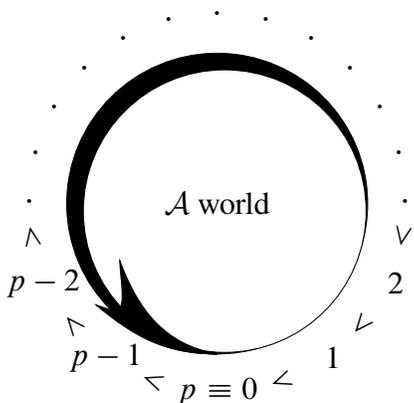


Figure 3.3: mod  $p$

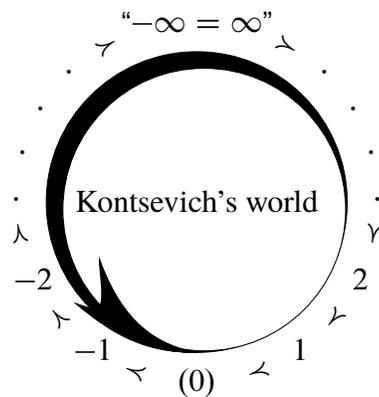


Figure 3.4: Kontsevich’s order

By this new order, the series expression of SMZVs are given as follows.

**Proposition 3.3.4** (Yasuda [26], Zagier). *For an index  $\mathbf{k}$  and a positive integer  $M$ , we set*

$$\zeta_{S,M}^*(\mathbf{k}) := \sum_{\substack{m_1 < \cdots < m_r \\ 0 < |m_1|, \dots, |m_r| < M}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}, \quad (3.3.2)$$

$$\zeta_{S,M}^{*,*}(\mathbf{k}) := \sum_{\mathbf{k}' \leq \mathbf{k}} \zeta_{S,M}^*(\mathbf{k}').$$

Then we have

$$\zeta_S^*(\mathbf{k}) = \lim_{M \rightarrow \infty} \zeta_{S,M}^*(\mathbf{k}) = \lim_{M \rightarrow \infty} \sum_{\substack{m_1 < \dots < m_r \\ 0 < |m_1|, \dots, |m_r| < M}} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}, \quad (3.3.3)$$

$$\zeta_S^{*,*}(\mathbf{k}) = \lim_{M \rightarrow \infty} \zeta_{S,M}^{*,*}(\mathbf{k}) = \lim_{M \rightarrow \infty} \sum_{\substack{m_1 \leq \dots \leq m_r \\ 0 < |m_1|, \dots, |m_r| < M}} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}. \quad (3.3.4)$$

*Proof.* First, by definition (3.3.2), we obtain

$$\begin{aligned} \zeta_{S,M}^*(\mathbf{k}) &= \sum_{i=0}^r \left( \sum_{\substack{m_1 < \dots < m_i \\ 0 < m_1, \dots, m_i \leq M}} \frac{1}{m_1^{k_1} \dots m_i^{k_i}} \right) \left( \sum_{\substack{m_{i+1} < \dots < m_r \\ -M \leq m_{i+1}, \dots, m_r < 0}} \frac{1}{m_{i+1}^{k_{i+1}} \dots m_r^{k_r}} \right) \\ &= \sum_{i=0}^r (-1)^{k_{i+1} + \dots + k_r} \zeta_M(k_1, \dots, k_i) \zeta_M(k_r, \dots, k_{i+1}). \end{aligned}$$

It is known [13] that

$$\zeta_M(\mathbf{k}) = \zeta^*(\mathbf{k}; \gamma + \log M) + O(M^{-1} \log^J M), \quad (3.3.5)$$

for some  $J$  as  $M \rightarrow \infty$ , where  $\gamma$  is Euler's constant defined in (2.5.2). By (3.3.5), we have

$$\begin{aligned} \zeta_{S,M}(\mathbf{k}) &= \sum_{i=0}^r (-1)^{k_{i+1} + \dots + k_r} \left( \zeta^*(k_1, \dots, k_i; \gamma + \log M) + O(M^{-1} \log^J M) \right) \\ &\quad \times \left( \zeta^*(k_r, \dots, k_{i+1}; \gamma + \log M) + O(M^{-1} \log^J M) \right) \\ &= \sum_{i=0}^r (-1)^{k_{i+1} + \dots + k_r} \zeta^*(k_1, \dots, k_i; \gamma + \log M) \zeta^*(k_r, \dots, k_{i+1}; \gamma + \log M) \\ &\quad + O(M^{-1} \log^{J+r} M) + O(M^{-2} \log^{2J} M) \quad (M \rightarrow \infty). \end{aligned}$$

As mentioned,  $\zeta_S^*(\mathbf{k})$  is independent of  $\gamma + \log M$ . Thus we have

$$\zeta_{S,M}(\mathbf{k}) = \zeta_S^*(\mathbf{k}) + O(M^{-1} \log^{J+r} M) + O(M^{-2} \log^{2J} M) \quad (M \rightarrow \infty). \quad (3.3.6)$$

Taking the limit  $M \rightarrow \infty$  in (3.3.6), we obtain (3.3.3).

The sum of (3.3.6) over all  $\mathbf{k}' \leq \mathbf{k}$ , implies

$$\zeta_{S,M}^{*,*}(\mathbf{k}) = \zeta_S^{*,*}(\mathbf{k}) + O(M^{-1} \log^{J+r} M) + O(M^{-2} \log^{2J} M) \quad (M \rightarrow \infty),$$

and hence (3.3.4). □

From the definition of  $\zeta_{S,M}^*(\mathbf{k})$ , the harmonic rule for  $\zeta_{S,M}^*(\mathbf{k})$  is shown by the same proof for  $\zeta_M(\mathbf{k} * \mathbf{l}) = \zeta_M(\mathbf{k})\zeta_M(\mathbf{l})$  by replacing the inequality  $m_1 < \dots < m_r$  with the new inequality  $m_1 < \dots < m_r$ . Thus

$$\zeta_{S,M}^*(\mathbf{k} * \mathbf{l}) = \zeta_{S,M}^*(\mathbf{k})\zeta_{S,M}^*(\mathbf{l}) \quad (3.3.7)$$

holds. So, by taking the limit of (3.3.7) as  $M \rightarrow \infty$ , we find that SMZVs satisfy the harmonic relation.

**Corollary 3.3.5.** *For any indices  $\mathbf{k}$  and  $\mathbf{l}$ , we have*

$$\zeta_S^*(\mathbf{k} * \mathbf{l}) = \zeta_S^*(\mathbf{k})\zeta_S^*(\mathbf{l}).$$

By understanding  $\zeta_S(\mathbf{k} \boxplus \mathbf{l})$  as the expansion of  $\zeta(\mathbf{k} \boxplus \mathbf{l})$  with formally replacing “ $\zeta$ ” with “ $\zeta_S$ ”, we have the following relation.

**Theorem 3.3.6.** *For two indices  $\mathbf{k}$  and  $\mathbf{l}$ , we have*

$$\zeta_S(\mathbf{k} \boxplus \mathbf{l}) = (-1)^{\text{wt}(\mathbf{l})}\zeta_S(\mathbf{k}, \bar{\mathbf{l}}), \quad (3.3.8)$$

where  $\bar{\mathbf{l}} = (l_s, \dots, l_1)$  for  $\mathbf{l} = (l_1, \dots, l_s)$ .

We refer the reader to [9] for a proof of Proposition 3.3.6. We remark that (3.3.8) holds without modulo  $\zeta(2)\mathcal{Z}$  and taking the limit  $M \rightarrow \infty$ :

$$\zeta_{S,M}^{\boxplus}(\mathbf{k} \boxplus \mathbf{l}) = (-1)^{\text{wt}(\mathbf{l})}\zeta_{S,M}^{\boxplus}(\mathbf{k}, \bar{\mathbf{l}}),$$

where

$$\zeta_{S,M}^{\boxplus}(\mathbf{k}) := \sum_{i=0}^r \sum_{\substack{0 < m_1 < \dots < m_i \\ m_{i+1} < \dots < m_r < 0 \\ m_i - m_{i+1} < M}} \frac{1}{m_1^{k_1} \dots m_r^{k_r}},$$

for an index  $\mathbf{k} = (k_1, \dots, k_r)$ , which is introduced in [21]. The authors of [21] showed

$$\zeta_S^{\boxplus}(\mathbf{k}) = \lim_{M \rightarrow \infty} \zeta_{S,M}^{\boxplus}(\mathbf{k}).$$

Note that specializing  $\mathbf{l} = \emptyset$  in (3.1.3) and (3.3.8), we have a relation for FMZVs and SMZVs which is called the reverse formula.

**Corollary 3.3.7.** *For an index  $\mathbf{k}$ , we have*

$$\zeta_{\mathcal{F}}(\mathbf{k}) = (-1)^{\text{wt}(\mathbf{k})}\zeta_{\mathcal{F}}(\bar{\mathbf{k}}), \quad (3.3.9)$$

$$\zeta_{\mathcal{F}}^*(\mathbf{k}) = (-1)^{\text{wt}(\mathbf{k})}\zeta_{\mathcal{F}}^*(\bar{\mathbf{k}}). \quad (3.3.10)$$

(3.3.10) is obtained by (3.3.9) and the definition of  $\zeta_{\mathcal{F}}^*(\mathbf{k})$  as follows:

$$\zeta_{\mathcal{F}}^*(\mathbf{k}) = \sum_{\mathbf{k}' \leq \mathbf{k}} \zeta_{\mathcal{F}}(\mathbf{k}') = \sum_{\mathbf{k}' \leq \mathbf{k}} (-1)^{\text{wt}(\mathbf{k}')} \zeta_{\mathcal{F}}(\overline{\mathbf{k}'}) = (-1)^{\text{wt}(\mathbf{k})} \sum_{\mathbf{k}' \leq \overline{\mathbf{k}}} \zeta_{\mathcal{F}}(\mathbf{k}') = (-1)^{\text{wt}(\mathbf{k})} \zeta_{\mathcal{F}}^*(\overline{\mathbf{k}}).$$

**Theorem 3.3.8** (Symmetric sum formula,  $\mathcal{A}$ : Hoffman [11],  $\mathcal{S}$ : Murahara [20]). *Let  $(k_1, \dots, k_r)$  be any index, and let  $\mathfrak{S}_r$  be the symmetric group of degree  $r$ . Then we have*

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}}(k_{\sigma(1)}, \dots, k_{\sigma(r)}) &= 0, \\ \sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}}^*(k_{\sigma(1)}, \dots, k_{\sigma(r)}) &= 0. \end{aligned} \quad (3.3.11)$$

The following three relations for FMZVs and SMZVs are  $\mathcal{F}$ -analogues of those introduced in Section 2.6.

**Theorem 3.3.9.** *For  $k, r \in \mathbb{Z}_{>0}$  with  $k > r$ , we have*

$$\begin{aligned} \sum_{\mathbf{k} \in I_0(k, r, *)} (-1)^r \zeta_{\mathcal{F}}(\mathbf{k}) &= \left( \binom{k-1}{r-1} + (-1)^k \right) \mathfrak{Z}_{\mathcal{F}}(k), \\ \sum_{\mathbf{k} \in I_0(k, r, *)} \zeta_{\mathcal{F}}^*(\mathbf{k}) &= \left( \binom{k-1}{r-1} + (-1)^k \right) \mathfrak{Z}_{\mathcal{F}}(k). \end{aligned}$$

The sum formula for  $\mathcal{F}$  was conjectured by Kaneko in [14] and proved by Saito–Wakabayashi for  $\mathcal{A}$  and Murahara for  $\mathcal{S}$  in an extended form as follows.

**Theorem 3.3.10** ( $\mathcal{A}$ : Saito–Wakabayashi, [22],  $\mathcal{S}$ : Murahara, [20]). *For  $i, k, r \in \mathbb{Z}_{>0}$  with  $i \geq r \geq k-1$ , we have*

$$\begin{aligned} \sum_{\substack{\mathbf{k} \in I_0(k, r, *) \\ k_i \geq 2}} (-1)^r \zeta_{\mathcal{F}}(\mathbf{k}) &= (-1)^{i-1} \left( \binom{k-1}{n-i} + (-1)^r \binom{k-1}{i-1} \right) \mathfrak{Z}_{\mathcal{F}}(k), \\ \sum_{\substack{\mathbf{k} \in I_0(k, r, *) \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(\mathbf{k}) &= (-1)^{i-1} \left( \binom{k-1}{n-i} + (-1)^r \binom{k-1}{i-1} \right) \mathfrak{Z}_{\mathcal{F}}(k). \end{aligned}$$

We remark that the case  $i = 1$  in Theorem 3.3.10 reduces to Theorem 3.3.9.

**Theorem 3.3.11** (Aoki–Ohno’s relation,  $\mathcal{A}$ : Kaneko–Oyama–Saito [16]). *For a positive integers  $k$  and  $s$  with  $k \geq 2s$ , we have*

$$\sum_{\mathbf{k} \in I_0(k, *, s)} (-1)^{\text{dep}(\mathbf{k})} \zeta_{\mathcal{F}}(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1-2^{1-k}) \mathfrak{Z}_{\mathcal{F}}(k),$$

$$\sum_{\mathbf{k} \in I_0(k, *, s)} \zeta_{\mathcal{F}}^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1-2^{1-k}) \mathfrak{Z}_{\mathcal{F}}(k).$$

Aoki–Ohno’s relations for  $\mathcal{A}$  and  $\mathcal{S}$  were conjectured by Kaneko [17] and was proved for  $\mathcal{A}$  by Kaneko–Oyama–Saito [16]. However, the  $\mathcal{S}$ -analogue of these relations were not yet proved. In this thesis, Aoki–Ohno’s relations for  $\mathcal{S}$  are proved as a corollary of our main result.

**Theorem 3.3.12** (Generalized height-one duality,  $\mathcal{F}$ : Sakurada [25]). *For  $m, n, s \in \mathbb{Z}_{>0}$  with  $m, n \geq s \geq 1$ , we have*

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} (-1)^n \zeta_{\mathcal{F}}(\mathbf{k}) = (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} (-1)^m \zeta_{\mathcal{F}}(\mathbf{k}),$$

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_{\mathcal{F}}^*(\mathbf{k}) = (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_{\mathcal{F}}^*(\mathbf{k}).$$

Theorem 3.3.12 was conjectured by Kaneko, and was proved by Sakurada for both  $\mathcal{A}$  and  $\mathcal{S}$  in his master thesis and [25].

In the next chapter, we give proofs for the  $\mathcal{S}$ -version of Theorem 3.3.9, Theorem 3.3.11 and Theorem 3.3.12 as its of our Main Theorem.

# Chapter 4

## Main result

In this chapter, we show a congruence between symmetric multiple zeta-star values and multiple zeta-star values. To prove the result, we define a truncated polylogarithm-like sums and consider differential equations of a finite linear of these combination sums.

Secondly, we explain that Main Theorem directly leads to Aoki–Ohno’s relation, the sum formula and the generalized height-one duality for SMZ(S)V.

### 4.1 Main result and its corollaries

In this section, we establish an identity for SMZSVs and MZSVs. For this purpose, we define  $P_{k,r,s} \in \mathbb{Q}[\zeta(3), \zeta(5), \zeta(7), \dots]$  as

$$\begin{aligned} & \frac{1}{xy - z^2} \left( \exp \left( \sum_{k,l,m \geq 0} (-1)^{k+1} \frac{\zeta(2k + 2l + 2m + 3)}{k + 1} \right. \right. \\ & \quad \left. \left. \times \binom{k + l + 2m + 1}{k, l, 2m + 1} (xy - z^2)^{k+1} (xy)^l (x - y)^{2m+1} \right) - 1 \right) \\ & \qquad \qquad \qquad = \sum_{k,r,s \geq 0} P_{k,r,s} x^{k-r-s} y^{r-s} z^{2s-2} \in \mathcal{Z}[[x, y, z]]. \end{aligned}$$

**Theorem 4.1.1** (Main Theorem). *We have*

$$(-1)^{k-1} \sum_{\mathbf{k} \in I_0(k,r,s)} (-1)^r \zeta_S^*(\mathbf{k}) = \sum_{\mathbf{k} \in I_0(k,r,s)} \zeta^*(\mathbf{k}) + P_{k,r,s} \pmod{\zeta(2)\mathcal{Z}}, \quad (4.1.1)$$

$$\sum_{\mathbf{k} \in I_0(k,r,s)} \zeta_S^*(\mathbf{k}) = \sum_{\mathbf{k} \in I_0(k,r,s)} \zeta^*(\mathbf{k}) + P_{k,r,s} \pmod{\zeta(2)\mathcal{Z}}. \quad (4.1.2)$$

From Main Theorem and sum formulas for MZSVs, we easily obtain the corresponding sum formulas for SMZSVs. In particular, as mentioned in Theorem 3.3.11, we give a proof of Aoki–Ohno’s relations for SMZVs for the first time.

**Corollary 4.1.2** ([6], cf. Theorem 3.3.11 [1, 16]). *For  $k, s \in \mathbb{Z}_{>0}$  with  $k \geq 2s$ ,*

$$\sum_{\mathbf{k} \in I_0(k, *, s)} (-1)^{\text{dep}(\mathbf{k})} \zeta_S(\mathbf{k}) = \sum_{\mathbf{k} \in I_0(k, *, s)} \zeta_S^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \zeta(k) \pmod{\zeta(2)\mathcal{Z}}$$

*holds.*

**Corollary 4.1.3** (Theorem 3.3.9 [20, 22]). *For  $k, r \in \mathbb{Z}_{>0}$  with  $k > r$ ,*

$$\sum_{\mathbf{k} \in I_0(k, r, *)} (-1)^r \zeta_S(\mathbf{k}) = \sum_{\mathbf{k} \in I_0(k, r, *)} \zeta_S^*(\mathbf{k}) = \left( \binom{k-1}{r-1} + (-1)^r \right) \zeta(k) \pmod{\zeta(2)\mathcal{Z}}$$

*holds.*

**Corollary 4.1.4** (Theorem 3.3.12 [25]). *For  $m, n, s \in \mathbb{Z}_{>0}$  with  $m, n \geq s \geq 1$ ,*

$$\begin{aligned} (-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_S^*(\mathbf{k}) &= (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_S^*(\mathbf{k}), \\ (-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} (-1)^n \zeta_S(\mathbf{k}) &= (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} (-1)^m \zeta_S(\mathbf{k}). \end{aligned}$$

*holds.*

### 4.1.1 Preparation for proofs

To prove Main Theorem, we define a truncated polylog-like sum.

**Definition 4.1.5.** For an index  $\mathbf{k} = (k_1, \dots, k_r)$  and a positive integer  $M$ , we define

$$\mathcal{L}_M^*(\mathbf{k}; t) = \mathcal{L}_M^*(k_1, \dots, k_r; t) := \sum_{\substack{m_1 \leq \dots \leq m_r \\ 0 < |m_1|, \dots, |m_{r-1}| \leq M \\ -M \leq m_r < 0}} \frac{t^{-m_r}}{m_1^{k_1} \dots m_r^{k_r}} \in \mathbb{Q}[t].$$

We remark that  $\mathcal{L}_M^*(\mathbf{k}; t)$  is a polynomial in  $t$  of order at most  $M$  without a constant term and the order “ $<$ ” denotes Kontsevich’s order which is defined in (3.3.1). We note several properties for this new polylogarithm.

**Lemma 4.1.6.** For an index  $\mathbf{k}$ , we have

$$\mathcal{L}_M^*(\mathbf{k}; 0) = 0, \quad (4.1.3)$$

$$\mathcal{L}_M^*(\mathbf{k}; 1) = \zeta_{S,M}^{*,*}(\mathbf{k}) - \zeta_M^*(\mathbf{k}). \quad (4.1.4)$$

*Proof.* (4.1.3) is easily led by definition.

By definition, we have

$$\begin{aligned} \mathcal{L}_M^*(\mathbf{k}; 1) + \zeta_M^*(\mathbf{k}) &= \sum_{\substack{m_1 \leq \dots \leq m_r \\ 0 < |m_1|, \dots, |m_r| \leq M}} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \\ &= \zeta_{S,M}^{*,*}(\mathbf{k}), \end{aligned}$$

which implies (4.1.4), and completes the proof.  $\square$

Now, it is clear from (4.1.4) that  $\mathcal{L}_M^*(\mathbf{k}; 1)$  describes the difference between  $\zeta_{S,M}^{*,*}(\mathbf{k})$  and  $\zeta_M^*(\mathbf{k})$ . Next, we derive differential equations for  $\mathcal{L}_M^*(\mathbf{k}; t)$ .

**Lemma 4.1.7.** For an index  $(k_1, \dots, k_r)$ , we have

$$\frac{d}{dt} \mathcal{L}_M^*(k_1, \dots, k_r; t) = \begin{cases} -\frac{1}{t} \mathcal{L}_M^*(k_1, \dots, k_{r-1}, k_r - 1; t) & (k_r > 1), \\ \frac{1}{1-t} \left( \mathcal{L}_M^*(k_1, \dots, k_{r-1}; t) - \zeta_{S,M}^{*,*}(k_1, \dots, k_{r-1}) + t^M \zeta_M^*(k_1, \dots, k_{r-1}) \right) & (k_r = 1, r \geq 2), \\ -\frac{1-t^M}{1-t} & (k_r = 1, r = 1). \end{cases}$$

*Proof.* If  $k_r > 1$ , then we have the following equation by termwise differentiation:

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_M^*(k_1, \dots, k_r; t) &= \frac{d}{dt} \left( \sum_{\substack{m_1 \leq \dots \leq m_r \\ 0 < |m_1|, \dots, |m_{r-1}| \leq M \\ -M \leq m_r < 0}} \frac{t^{-m_r}}{m_1^{k_1} \dots m_r^{k_r}} \right) \\ &= -\frac{1}{t} \left( \sum_{\substack{m_1 \leq \dots \leq m_r \\ 0 < |m_1|, \dots, |m_{r-1}| \leq M \\ -M \leq m_r < 0}} \frac{t^{-m_r}}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}} m_r^{k_r-1}} \right) \\ &= -\frac{1}{t} \mathcal{L}_M^*(k_1, \dots, k_{r-1}, k_r - 1; t), \end{aligned}$$

which yields the first equality.

If  $k_r = 1$  with  $r \geq 2$ , then similarly we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{L}_M^*(k_1, \dots, k_{r-1}, 1; t) &= -\frac{1}{t} \left( \sum_{\substack{m_1 \leq \dots \leq m_r \\ 0 < |m_1|, \dots, |m_{r-1}| \leq M \\ -M \leq m_r < 0}} \frac{t^{-m_r}}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \right) \\
&= -\frac{1}{t} \left( \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ -M \leq m_{r-1} < 0}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \sum_{m_r = m_{r-1}}^{-1} t^{-m_r} \right. \\
&\quad \left. + \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ 0 < m_{r-1} \leq M}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \sum_{m_r = -M}^{-1} t^{-m_r} \right).
\end{aligned}$$

Here, the first term is

$$\begin{aligned}
&-\frac{1}{t} \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ -M \leq m_{r-1} < 0}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \cdot \frac{t(1 - t^{-m_{r-1}})}{1 - t} \\
&= -\frac{1}{1 - t} \left( \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ -M \leq m_{r-1} < 0}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} - \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ -M \leq m_{r-1} < 0}} \frac{t^{-m_{r-1}}}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \right), \quad (4.1.5)
\end{aligned}$$

while the second term is

$$\begin{aligned}
&-\frac{1}{t} \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ 0 < m_{r-1} \leq M}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \cdot \frac{t(1 - t^M)}{1 - t} \\
&= -\frac{1}{1 - t} \left( \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ 0 < m_{r-1} \leq M}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} - t^M \sum_{\substack{m_1 \leq \dots \leq m_{r-1} \\ 0 < |m_1|, \dots, |m_{r-2}| \leq M \\ 0 < m_{r-1} \leq M}} \frac{1}{m_1^{k_1} \dots m_{r-1}^{k_{r-1}}} \right). \quad (4.1.6)
\end{aligned}$$

Hence, by (4.1.5) and (4.1.6), we have

$$\begin{aligned} & \frac{d}{dt} \mathcal{L}_M^*(k_1, \dots, k_{r-1}, 1; t) \\ &= \frac{1}{1-t} \left( \mathcal{L}_M^*(k_1, \dots, k_{r-1}; t) - \zeta_{S,M}^{*,*}(k_1, \dots, k_{r-1}) + t^M \zeta_M^*(k_1, \dots, k_{r-1}) \right), \end{aligned}$$

which gives the second equality.

Lastly, if  $k_r = 1$  and  $r = 1$ , then we have

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_M^*(1; t) &= -\frac{1}{t} \sum_{m=1}^M t^m \\ &= -\frac{1-t^M}{1-t}. \end{aligned}$$

□

Let us introduce some finite sums concerning truncated MZ(S)V's and SMZ(S)V's. For  $k, r, s \in \mathbb{Z}$ , we set

$$\begin{aligned} X_{0,M}^*(k, r, s) &:= \sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_M^*(\mathbf{k}), \\ X_{0,S,M}^*(k, r, s) &:= \sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_{S,M}^{*,*}(\mathbf{k}). \end{aligned}$$

Here and hereafter, the empty sum is understood as 0. For  $k, r, s \in \mathbb{Z}$  with  $(k, r, s) \neq (0, 0, 0)$ , we set

$$\begin{aligned} X_M^*(k, r, s) &:= \sum_{\mathbf{k} \in I(k, r, s)} \zeta_M^*(\mathbf{k}), \\ X_{S,M}^*(k, r, s) &:= \sum_{\mathbf{k} \in I(k, r, s)} \zeta_{S,M}^{*,*}(\mathbf{k}), \end{aligned}$$

and

$$\begin{aligned} X_M^*(0, 0, 0) &:= 1, \\ X_{S,M}^*(0, 0, 0) &:= 1. \end{aligned}$$

Furthermore, for  $k, r, s \in \mathbb{Z}$ , we define

$$D_{0,M}^*(k, r, s; t) := \sum_{\mathbf{k} \in I_0(k, r, s)} \mathcal{L}_M^*(\mathbf{k}; t),$$

$$D_M^*(k, r, s; t) := \sum_{\mathbf{k} \in I(k, r, s)} \mathcal{L}_M^*(\mathbf{k}; t).$$

The functions  $D_{0,M}^*(k, r, s; t)$  and  $D_M^*(k, r, s; t)$  satisfy the following differential equations with respect to  $t$ .

**Lemma 4.1.8.** *We have*

$$\frac{d}{dt} D_{0,M}^*(k, r, s; t) = -\frac{1}{t} \left( D_M^*(k-1, r, s-1; t) - D_{0,M}^*(k-1, r, s-1; t) + D_{0,M}^*(k-1, r, s; t) \right), \quad (4.1.7)$$

$$\begin{aligned} & \frac{d}{dt} \left( D_M^*(k, r, s; t) - D_{0,M}^*(k, r, s; t) \right) \\ &= \frac{1}{1-t} \left( D_M^*(k-1, r-1, s; t) - X_{S,M}^*(k-1, r-1, s) + t^M X_M^*(k-1, r-1, s) \right). \end{aligned} \quad (4.1.8)$$

*Proof.* By use of the decomposition

$$I_0(k, r, s) = \{\mathbf{k} \mid \mathbf{k} \in I_0(k, r, s), k_r = 2\} \sqcup \{\mathbf{k} \mid \mathbf{k} \in I_0(k, r, s), k_r > 2\},$$

and Lemma 4.1.7, we obtain

$$\begin{aligned} \frac{d}{dt} D_{0,M}^*(k, r, s; t) &= \frac{d}{dt} \left[ \left( \sum_{\substack{\mathbf{k} \in I_0(k, r, s) \\ k_r=2}} + \sum_{\substack{\mathbf{k} \in I_0(k, r, s) \\ k_r>2}} \right) \mathcal{L}_M^*(\mathbf{k}; t) \right] \\ &= -\frac{1}{t} \left( \sum_{\substack{\mathbf{k} \in I(k-1, r, s-1) \\ k_r=1}} \mathcal{L}_M^*(\mathbf{k}; t) + \sum_{\mathbf{k} \in I_0(k-1, r, s)} \mathcal{L}_M^*(\mathbf{k}; t) \right) \\ &= -\frac{1}{t} \left( D_M^*(k-1, r, s-1; t) - D_{0,M}^*(k-1, r, s-1; t) + D_{0,M}^*(k-1, r, s; t) \right), \end{aligned}$$

which implies (4.1.7).

Next, we derive (4.1.8). For  $(k, r, s) \neq (1, 1, 0)$ , we have

$$\begin{aligned}
\frac{d}{dt}(D_M^*(k, r, s; t) - D_{0,M}^*(k, r, s; t)) &= \frac{d}{dt} \left( \sum_{\substack{\mathbf{k} \in I(k, r, s) \\ k_r = 1}} \mathcal{L}_M^*(\mathbf{k}; t) \right) \\
&= \frac{1}{1-t} \left( \sum_{\mathbf{k} \in I(k-1, r-1, s)} \left( \mathcal{L}_M^*(\mathbf{k}; t) - \zeta_{S,M}^{*,*}(\mathbf{k}) + t^M \zeta_M^*(\mathbf{k}) \right) \right) \\
&= \frac{1}{1-t} \left( D_M^*(k-1, r-1, s; t) \right. \\
&\quad \left. - X_{S,M}^*(k-1, r-1, s) + t^M X_M^*(k-1, r-1, s) \right).
\end{aligned}$$

For  $(k, r, s) = (1, 1, 0)$ , we have

$$\begin{aligned}
\frac{d}{dt}(D_M^*(1, 1, 0; t) - D_{0,M}^*(1, 1, 0; t)) &= \frac{d}{dt}(\mathcal{L}_M^*(1; t)) \\
&= -\frac{1-t^M}{1-t} \\
&= \frac{1}{1-t} \left( D_M^*(0, 0, 0; t) - X_{S,M}^*(0, 0, 0) + t^M X_M^*(0, 0, 0) \right).
\end{aligned}$$

□

We denote by  $\Phi_{0,M}^*, \Phi_M^*, \Phi_{0,S,M}^*, \Phi_{S,M}^* \in \mathbb{R}[[x, y, z]]$  and  $\Delta_{0,M}^*(t), \Delta_M^*(t) \in \mathbb{Q}[t][[x, y, z]]$

their generating functions:

$$\begin{aligned}
\Phi_{0,M}^* &= \Phi_{0,M}^*(x, y, z) := \sum_{k,r,s \geq 0} X_{0,M}^*(k, r, s) x^{k-r-s} y^{r-s} z^{2s-2}, \\
\Phi_M^* &= \Phi_M^*(x, y, z) := \sum_{k,r,s \geq 0} X_M^*(k, r, s) x^{k-r-s} y^{r-s} z^{2s}, \\
\Phi_{0,S,M}^* &= \Phi_{0,S,M}^*(x, y, z) := \sum_{k,r,s \geq 0} X_{0,S,M}^*(k, r, s) x^{k-r-s} y^{r-s} z^{2s-2}, \\
\Phi_{S,M}^* &= \Phi_{S,M}^*(x, y, z) := \sum_{k,r,s \geq 0} X_{S,M}^*(k, r, s) x^{k-r-s} y^{r-s} z^{2s}, \\
\Delta_{0,M}^*(t) &= \Delta_{0,M}^*(x, y, z; t) := \sum_{k,r,s \geq 0} D_{0,M}^*(k, r, s; t) x^{k-r-s} y^{r-s} z^{2s-2}, \\
\Delta_M^*(t) &= \Delta_M^*(x, y, z; t) := \sum_{k,r,s \geq 0} D_M^*(k, r, s; t) x^{k-r-s} y^{r-s} z^{2s},
\end{aligned}$$

where  $\mathbb{R}[[x, y, z]]$  and  $\mathbb{Q}[t][[x, y, z]]$  are equipped with the product topology. Furthermore, we denote by  $\Phi_0^*$ ,  $\Phi_{0,S}^*$ ,  $\Phi_S^*$  and  $\Delta_0^*$

$$\begin{aligned}
\Phi_0^* &:= \lim_{M \rightarrow \infty} \Phi_{0,M}^*, \\
\Phi_{0,S}^* &:= \lim_{M \rightarrow \infty} \Phi_{0,S,M}^*, \\
\Phi_S^* &:= \lim_{M \rightarrow \infty} \Phi_{S,M}^*, \\
\Delta_0^* &:= \lim_{M \rightarrow \infty} \Delta_{0,M}^*(1).
\end{aligned}$$

We state several facts obtained by applying Lemma 4.1.6 to the generating functions. By the symmetric sum formula for SMZSVs (3.3.11), we see

$$\begin{aligned}
\Phi_S^* &= \sum_{\substack{k,r,s \geq 0 \\ (k,r,s) \neq (0,0,0)}} \sum_{\mathbf{k} \in I(k,r,s)} \zeta_S^{*,*}(\mathbf{k}) x^{k-r-s} y^{r-s} z^{2s} + 1 \\
&\equiv 1 \pmod{\zeta(2)\mathcal{Z}}.
\end{aligned} \tag{4.1.9}$$

By (4.1.3), we have the following equation for all  $k, r, s \in \mathbb{Z}$ :

$$D_{0,M}^*(k, r, s; 0) = 0,$$

and hence

$$\Delta_{0,M}^*(0) = 0. \tag{4.1.10}$$

Moreover, by (4.1.4), we have

$$D_{0,M}^*(k, r, s; 1) = X_{0,S,M}^*(k, r, s) - X_{0,M}^*(k, r, s).$$

Thus

$$\begin{aligned} \Delta_0^* &= \Phi_{0,S}^* - \Phi_0^* \\ &= \sum_{k,r,s \geq 0} \sum_{\mathbf{k} \in I_0(k,r,s)} \left( \zeta_S^{*,*}(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) x^{k-r-s} y^{r-s} z^{2s-2}. \end{aligned} \quad (4.1.11)$$

We rephrase Lemma 4.1.8 in terms of the generating functions  $\Delta_{0,M}^*(t)$  and  $\Delta_M^*(t)$ .

**Lemma 4.1.9.** *We have*

$$\frac{d\Delta_{0,M}^*(t)}{dt} = -\frac{1}{yt} \left( \Delta_M^*(t) + (xy - z^2) \Delta_{0,M}^*(t) \right), \quad (4.1.12)$$

$$\frac{d}{dt} \left( \Delta_M^*(t) - z^2 \Delta_{0,M}^*(t) \right) = \frac{y}{1-t} \Delta_M^*(t) - \frac{y}{1-t} \Phi_{S,M}^* + \frac{yt^M}{1-t} \Phi_M^*. \quad (4.1.13)$$

*Proof.* By (4.1.7), we have

$$\begin{aligned} \frac{d\Delta_{0,M}^*(t)}{dt} &= -\frac{1}{t} \left( \sum_{k,r,s \geq 0} D_M^*(k-1, r, s-1; t) x^{k-r-s} y^{r-s} z^{2s-2} \right) \\ &\quad + \frac{1}{t} \left( \sum_{k,r,s \geq 0} D_{0,M}^*(k-1, r, s-1; t) x^{k-r-s} y^{r-s} z^{2s-2} \right) \\ &\quad - \frac{1}{t} \left( \sum_{k,r,s \geq 0} D_{0,M}^*(k-1, r, s; t) x^{k-r-s} y^{r-s} z^{2s-2} \right) \\ &= -\frac{1}{yt} \left( \sum_{k,r,s \geq 0} D_M^*(k, r, s; t) x^{k-r-s} y^{r-s} z^{2s} \right) \\ &\quad + \frac{z^2}{yt} \left( \sum_{k,r,s \geq 0} D_{0,M}^*(k, r, s; t) x^{k-r-s} y^{r-s} z^{2s-2} \right) \\ &\quad - \frac{x}{t} \left( \sum_{k,r,s \geq 0} D_{0,M}^*(k, r, s; t) x^{k-r-s} y^{r-s} z^{2s-2} \right). \end{aligned}$$

Hence, we have

$$\frac{d\Delta_{0,M}^*(t)}{dt} = -\frac{1}{yt} \left( \Delta_M^*(t) + (xy - z^2) \Delta_{0,M}^*(t) \right),$$

which gives (4.1.12).

By using (4.1.8), we have

$$\begin{aligned}
\frac{d}{dt} \left( \Delta_M^*(t) - z^2 \Delta_{0,M}^*(t) \right) &= \frac{d}{dt} \left( \sum_{k,r,s \geq 0} \left( D_M^*(k,r,s;t) - D_{0,M}^*(k,r,s;t) \right) x^{k-r-s} y^{r-s} z^{2s} \right) \\
&= \frac{1}{1-t} \left( \sum_{k,r,s \geq 0} D_M^*(k-1,r-1,s;t) x^{k-r-s} y^{r-s} z^{2s} \right. \\
&\quad \left. - \sum_{k,r,s \geq 0} X_{S,M}^*(k-1,r-1,s) x^{k-r-s} y^{r-s} z^{2s} \right. \\
&\quad \left. + t^M \sum_{k,s \geq 0} X_M^*(k-1,r-1,s) x^{k-r-s} y^{r-s} z^{2s} \right) \\
&= \frac{y}{1-t} \left( \sum_{k,r,s \geq 0} D_M^*(k,r,s;t) x^{k-r-s} y^{r-s} z^{2s} \right. \\
&\quad \left. - \sum_{k,r,s \geq 0} X_{S,M}^*(k,r,s;t) x^{k-r-s} y^{r-s} z^{2s} \right. \\
&\quad \left. + t^M \sum_{k,r,s \geq 0} X_M^*(k,r,s;t) x^{k-r-s} y^{r-s} z^{2s} \right) \\
&= \frac{y}{1-t} \Delta_M^*(t) - \frac{y}{1-t} \Phi_{S,M}^* + \frac{y t^M}{1-t} \Phi_M^*,
\end{aligned}$$

which gives (4.1.13) and completes the proof.  $\square$

**Proposition 4.1.10.** *The generating function  $\Delta_{0,M}^*(t)$  satisfies the following differential equation:*

$$t(1-t) \frac{d^2 \Delta_{0,M}^*(t)}{dt^2} + \left( (1-t)(1+x) - ty \right) \frac{d \Delta_{0,M}^*(t)}{dt} - (xy - z^2) \Delta_{0,M}^*(t) = \Phi_{S,M}^* - t^M \Phi_M^*. \quad (4.1.14)$$

Furthermore, the solution of (4.1.14) around  $t = 0$  with the boundary condition (4.1.10) is uniquely given by

$$\Delta_{0,M}^*(t) = \Phi_{S,M}^* u_1(t) - \Phi_M^* u_{2,M}(t), \quad (4.1.15)$$

where

$$u_1(t) = \frac{1}{\alpha\beta} \left( {}_2F_1 \left( \begin{matrix} \alpha, \beta \\ x+1 \end{matrix}; t \right) - 1 \right), \quad (4.1.16)$$

$$u_{2,M}(t) = \frac{t^M}{(\alpha+M)(\beta+M)} \left( {}_3F_2 \left( \begin{matrix} \alpha+M, \beta+M, 1 \\ x+M+1, M+1 \end{matrix}; t \right) - 1 \right), \quad (4.1.17)$$

with

$$\alpha = \frac{x + y + \sqrt{(x - y)^2 + 4z^2}}{2}, \quad \beta = \frac{x + y - \sqrt{(x - y)^2 + 4z^2}}{2}.$$

*Proof.* To obtain the differential equation (4.1.14), we calculate the second derivative of  $\Delta_{0,M}^*(t)$  and eliminate  $\Delta_M^*(t)$ . By Lemma 4.1.9, we have

$$\begin{aligned} \frac{d^2 \Delta_{0,M}^*(t)}{dt^2} &= -\frac{1}{t} \left( -\frac{1}{yt} \left( \Delta_M^*(t) + (xy - z^2) \Delta_{0,M}^*(t) \right) \right) \\ &\quad - \frac{1}{yt} \left( \frac{y}{1-t} \Delta_M^*(t) - \frac{y}{1-t} \Phi_{S,M}^* + \frac{yt^M}{1-t} \Phi_M^* \right) - \frac{x}{t} \frac{d\Delta_{0,M}^*(t)}{dt} \\ &= -\frac{1}{t} (1+x) \frac{d\Delta_{0,M}^*(t)}{dt} + \frac{y}{1-t} \left( -\frac{1}{yt} \left( \Delta_M^*(t) + (xy - z^2) \Delta_{0,M}^*(t) \right) \right) \\ &\quad + \frac{xy - z^2}{t(1-t)} \Delta_{0,M}^*(t) + \frac{1}{t(1-t)} \Phi_{S,M}^* - \frac{t^M}{t(1-t)} \Phi_M^* \\ &= \left( -\frac{1}{t} (1+x) + \frac{y}{1-t} \right) \frac{d\Delta_{0,M}^*(t)}{dt} + \frac{xy - z^2}{t(1-t)} \Delta_{0,M}^*(t) + \frac{1}{t(1-t)} \Phi_{S,M}^* - \frac{t^M}{t(1-t)} \Phi_M^*. \end{aligned}$$

Multiplying both sides of the equation by  $t(1-t)$ , we obtain (4.1.14).

Next, we construct the power-series solution of (4.1.14) with the boundary condition (4.1.10). The differential equation determines the coefficients recursively and yields (4.1.15) with

$$u_1(t) = \sum_{n=0}^{\infty} a_n t^n, \quad u_{2,M}(t) = \sum_{n=0}^{\infty} b_{n,M} t^n$$

and

$$a_n = \begin{cases} 0 & (n = 0), \\ \frac{\prod_{j=1}^{n-1} \left( (x+j)(y+j) - z^2 \right)}{n!(x+1)_n} & (n \geq 1), \end{cases}$$

$$b_{n,M} = \begin{cases} 0 & (0 \leq n \leq M), \\ \frac{\prod_{j=1}^{n-M-1} \left( (x+M+j)(y+M+j) - z^2 \right)}{(x+M+1)_{n-M} (M+1)_{n-M}} & (n > M). \end{cases}$$

Since  $\alpha$  and  $\beta$  satisfy

$$(x + Y)(y + Y) - z^2 = (\alpha + Y)(\beta + Y),$$

we rewrite  $a_n$  and  $b_{n,M}$  as

$$a_n = \begin{cases} 0 & (n = 0), \\ \frac{(\alpha + 1)_{n-1}(\beta + 1)_{n-1}}{n!(x + 1)_n} & (n \geq 1), \end{cases}$$

$$b_{n,M} = \begin{cases} 0 & (0 \leq n \leq M), \\ \frac{(\alpha + M + 1)_{n-M-1}(\beta + M + 1)_{n-M-1}}{(x + M + 1)_{n-M}(M + 1)_{n-M}} & (n > M), \end{cases}$$

namely

$$\begin{aligned} u_1(t) &= \frac{1}{\alpha\beta} \left( \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(x + 1)_n} t^n - 1 \right) \\ &= \frac{1}{\alpha\beta} \left( {}_2F_1 \left( \begin{matrix} \alpha, \beta \\ x + 1 \end{matrix}; t \right) - 1 \right), \end{aligned} \quad (4.1.18)$$

and

$$\begin{aligned} u_{2,M}(t) &= \frac{t^M}{(\alpha + M)(\beta + M)} \sum_{n=1}^{\infty} \frac{(\alpha + M)_n(\beta + M)_n(1)_n}{n!(x + M + 1)_n(M + 1)_n} t^n \\ &= \frac{t^M}{(\alpha + M)(\beta + M)} \left( {}_3F_2 \left( \begin{matrix} \alpha + M, \beta + M, 1 \\ x + M + 1, M + 1 \end{matrix}; t \right) - 1 \right). \end{aligned} \quad (4.1.19)$$

□

Lastly, we show several lemmas. The second lemma and (4.1.2) lead to (4.1.1) and the third lemma is needed in the proof of Main Theorem.

**Lemma 4.1.11.** *For a function  $f : (\mathbb{Z} \setminus \{0\})^r \rightarrow \mathbb{C}$ , we obtain*

$$\sum_{i=0}^r (-1)^i \sum_{\substack{m_1 < \dots < m_i \\ m_{i+1} \geq \dots \geq m_r \\ 0 < |m_1|, \dots, |m_r| \leq M}} f(m_1, \dots, m_r) = 0.$$

*Proof.* Let  $D = \{(m_1, \dots, m_r) \in \mathbb{Z}^r \mid 0 < |m_1|, \dots, |m_r| < M\}$  and for  $0 \leq i \leq r$ , let

$$D_i = \{(m_1, \dots, m_r) \in D \mid m_1 < \dots < m_i, m_{i+1} \geq \dots \geq m_r\}.$$

For  $1 \leq i \leq r-1$ , we decompose  $D_i = E_i \sqcup F_i$  with

$$E_i = \{(m_1, \dots, m_r) \in D \mid m_1 < \dots < m_i \geq m_{i+1} \geq \dots \geq m_r\},$$

$$F_i = \{(m_1, \dots, m_r) \in D \mid m_1 < \dots < m_i < m_{i+1} \geq \dots \geq m_r\}$$

and set  $E_0 = \emptyset$ ,  $F_0 = D_0$  and  $E_r = D_r$ ,  $F_r = \emptyset$ . Then we see that  $E_{i+1} = F_i$  and the telescoping sum vanishes.  $\square$

**Lemma 4.1.12** (cf. [16, 20]). *We have*

$$\sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) = (-1)^{k-1} \sum_{\mathbf{k} \in I_0(k, r, s)} (-1)^r \zeta_{\mathcal{S}}(\mathbf{k}).$$

*Proof.* By Lemma 4.1.11, we have

$$\sum_{i=0}^r (-1)^i \zeta_{\mathcal{S}}(k_1, \dots, k_i) \zeta_{\mathcal{S}}^*(k_r, \dots, k_{i+1}) = 0.$$

Taking the sum of the equation above over all  $\mathbf{k} \in I_0(k, r, s)$  and separating the terms corresponding to  $i = 0$  and  $i = r$ , we have

$$\begin{aligned} & \sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) \\ & + \sum_{\substack{k'+k''=k \\ r'+r''=r \\ s'+s''=s}} \left( \sum_{\mathbf{k}' \in I_0(k', r', s')} (-1)^{r'} \zeta_{\mathcal{S}}(\overline{\mathbf{k}'}) \right) \left( \sum_{\mathbf{k}'' \in I(k'', r'', s'')} \zeta_{\mathcal{S}}^*(\mathbf{k}'') \right) \\ & + (-1)^k \sum_{\mathbf{k} \in I_0(k, r, s)} (-1)^r \zeta_{\mathcal{S}}(\mathbf{k}) = 0, \end{aligned}$$

where  $\overline{\mathbf{k}'}$  is the reversal of  $\mathbf{k}'$  and we used (3.3.9) in the third term. The middle term vanishes because of the symmetric sum formula for SMZSVs, which gives the assertion.  $\square$

Here, we state the generalized hypergeometric function  ${}_3F_2(A_1, A_2, A_3; B_1, B_2; t)$ , which is a special case with  $p = 3$  and  $q = 2$  in (2.6.9), and give a transformation formula of this function.

**Lemma 4.1.13** ([4], p. 98, Example 7). *For complex numbers  $A_1, A_2, A_3, B_1$  and  $B_2$  with  $\operatorname{Re} B_1 + B_2 - A_1 - A_2 - A_3 > 0$  and  $\operatorname{Re} B_2 - A_3 > 0$ , we have*

$${}_3F_2\left(\begin{matrix} A_1, A_2, A_3 \\ B_1, B_2 \end{matrix}; 1\right) = \frac{\Gamma(B_2)\Gamma(B_1 + B_2 - A_1 - A_2 - A_3)}{\Gamma(B_2 - A_3)\Gamma(B_1 + B_2 - A_1 - A_2)} {}_3F_2\left(\begin{matrix} B_1 - A_1, B_1 - A_2, A_3 \\ B_1, B_1 + B_2 - A_1 - A_2 \end{matrix}; 1\right). \quad (4.1.20)$$

## 4.2 Proof of main result

In this section, we describe the proof of our Main Theorem. (4.1.1) is easily led from (4.1.2) and Lemma 4.1.12, so it is enough to just prove (4.1.2).

*Proof of Main Theorem* Put  $t = 1$ . First, we show the second term of (4.1.15) vanishes as  $M \rightarrow \infty$ . By (4.1.19), we have

$$u_{2,M}(1) = \frac{1}{(\alpha + M)(\beta + M)} \left( {}_3F_2\left(\begin{matrix} \alpha + M, \beta + M, 1 \\ x + M + 1, M + 1 \end{matrix}; 1\right) - 1 \right).$$

Applying the transformation formula (4.1.20) to  ${}_3F_2(\alpha + M, \beta + M, 1; x + M + 1, M + 1; 1)$ , we have

$${}_3F_2\left(\begin{matrix} \alpha + M, \beta + M, 1 \\ x + M + 1, M + 1 \end{matrix}; 1\right) = \frac{M}{1 - y} {}_3F_2\left(\begin{matrix} x + 1 - \alpha, x + 1 - \beta, 1 \\ x + M + 1, 2 - y \end{matrix}; 1\right).$$

Here, the coefficient of  $x^l y^m z^n$  of the hypergeometric series above is given by

$$a_M(l, m, n) = \int_{C_x} \int_{C_y} \int_{C_z} {}_3F_2\left(\begin{matrix} x + 1 - \alpha, x + 1 - \beta, 1 \\ x + M + 1, 2 - y \end{matrix}; 1\right) \frac{dx dy dz}{x^{l+1} y^{m+1} z^{n+1}},$$

where  $C_x, C_y$  and  $C_z$  are circles whose radius is sufficiently small  $\epsilon > 0$  at the origin with a clockwise orientation on complex planes, respectively. Then we have

$$|a_M(l, m, n)| \leq \int_{C_x} \int_{C_y} \int_{C_z} \left| {}_3F_2\left(\begin{matrix} x + 1 - \alpha, x + 1 - \beta, 1 \\ x + M + 1, 2 - y \end{matrix}; 1\right) \right| \frac{|dx||dy||dz|}{|x^{l+1}||y^{m+1}||z^{n+1}|}.$$

Since

$$\begin{aligned} |(x + 1 + j - \alpha)(x + 1 + j - \beta)| &= |(x + 1 + j)^2 - (x + y)(x + 1 + j) + xy - z^2| \\ &= |(1 + j)^2 + (x - y)(1 + j) - z^2| \\ &\leq (1 + j)^2 + 2\epsilon(1 + j) + \epsilon^2 \\ &= (1 + j + \epsilon)^2 \end{aligned}$$

for  $0 \leq j \leq k$ , we have  $(x+1-\alpha)_k(x+1-\beta)_k = (1+\epsilon)_k(1+\epsilon)_k$ . Then for  $M \geq 1$ , we obtain

$$\begin{aligned} \left| {}_3F_2\left(\begin{matrix} x+1-\alpha, x+1-\beta, 1 \\ x+M+1, 2-y \end{matrix}; 1\right) \right| &\leq \sum_{k=0}^{\infty} \left| \frac{(x+1-\alpha)_k(x+1-\beta)_k(1)_k}{(x+M+1)_k(2-y)_k k!} \right| \\ &\leq {}_3F_2\left(\begin{matrix} 1+\epsilon, 1+\epsilon, 1 \\ 2-\epsilon, 2-\epsilon \end{matrix}; 1\right), \end{aligned}$$

and

$$|a_M(l, m, n)| \leq {}_3F_2\left(\begin{matrix} 1+\epsilon, 1+\epsilon, 1 \\ 2-\epsilon, 2-\epsilon \end{matrix}; 1\right).$$

By Raabe's test, we find that  ${}_3F_2(1+\epsilon, 1+\epsilon, 1; 2-\epsilon, 2-\epsilon; 1)$  absolutely converges. So, we have

$$a_M(l, m, n) = O(1) \quad (M \rightarrow \infty).$$

Hence, the coefficient of  $x^{k-r-s}y^{r-s}z^{2s}$  of  $u_{2,M}(1)$  is  $O(M^{-1})$ . Moreover, by the following estimation

$$\begin{aligned} \zeta_M^*(k_1, \dots, k_r) &= \sum_{1 \leq m_1 \leq \dots \leq m_r \leq M} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \\ &\leq \left( \sum_{m=1}^M \frac{1}{m} \right)^r \\ &\leq \left( 1 + \int_1^M \frac{1}{m} dm \right)^r \\ &= O(\log^r M) \quad (M \rightarrow \infty), \end{aligned}$$

we find the coefficient of  $x^{k-r-s}y^{r-s}z^{2s}$  of  $\Phi_M^*$  is  $O(\log^r M)$  and hence

$$\Phi_M^* u_{2,M}(1) \longrightarrow 0 \quad (M \rightarrow \infty). \quad (4.2.1)$$

Secondly, we show  $u_1(1)$  is an element of  $\mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots][[x, y, z]]$ . By (4.1.18) and the Gauss hypergeometric theorem, we obtain

$$u_1(1) = \frac{1}{\alpha\beta} \left( \frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\alpha)\Gamma(1+x-\beta)} - 1 \right).$$

Here, by using the formula (2.6.13) we have

$$\begin{aligned} \frac{\Gamma(1+x)\Gamma(1-y)}{\Gamma(1+x-\alpha)\Gamma(1+x-\beta)} &= \exp\left(\gamma(x-y) + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} (x^n + (-y)^n)\right) \\ &\quad \times \exp\left(-\gamma(2x-\alpha-\beta) - \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} ((x-\alpha)^n + (x-\beta)^n)\right) \\ &= \exp\left(\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} (x^n + (-y)^n - (x-\alpha)^n - (x-\beta)^n)\right). \end{aligned}$$

By the Taylor expansion of the logarithm, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x^n + (-y)^n - (x-\alpha)^n - (x-\beta)^n) t^n \\ &= \log\left(1 + \frac{(xy-z^2)t^2}{1+(x-y)t-xyt^2}\right) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{(xy-z^2)t^2}{1+(x-y)t-xyt^2}\right)^k \\ &= \sum_{k,l,m \geq 0} \left(\frac{(-1)^{k+m}}{k+1} \binom{k+l+m}{k,l,m}\right) (xy-z^2)^{k+1} (xy)^l (x-y)^m t^{2k+2l+m+2}, \end{aligned} \tag{4.2.2}$$

where we used  $\alpha + \beta = x + y$  and  $\alpha\beta = xy - z^2$ . Using (4.2.2), we rewrite

$$\begin{aligned} u_1(1) &= \frac{1}{xy-z^2} \left( \exp\left(\sum_{k,l,m \geq 0} (-1)^{k+m} \frac{\zeta(2k+2l+m+2)}{k+1} \right. \right. \\ &\quad \left. \left. \times \binom{k+l+m}{k,l,m} (xy-z^2)^{k+1} (xy)^l (x-y)^m\right) - 1 \right) \\ &\in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots][[x, y, z]]. \end{aligned} \tag{4.2.3}$$

By (4.1.9) and (4.2.1), we have

$$\begin{aligned} \Delta_0^* &= \lim_{M \rightarrow \infty} \Delta_{0,M}^*(1) = \Phi_S^* u_1(1) \\ &\equiv u_1(1) \pmod{\zeta(2)\mathcal{Z}}, \end{aligned}$$

and by (4.1.11) and (4.2.3), we finally obtain

$$\sum_{\mathbf{k} \in I_0(k,r,s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) = \sum_{\mathbf{k} \in I_0(k,r,s)} \zeta^*(\mathbf{k}) + P_{k,r,s} \pmod{\zeta(2)\mathcal{Z}},$$

which is the first equality (4.1.2). Lastly, (4.1.2) and Lemma 4.1.12 easily lead to (4.1.1), which completes the proof of Main Theorem.  $\square$

### 4.3 Proofs of corollaries

As announced in Section 3.3, we give a proof for  $\mathcal{S}$ -analogues of Aoki–Ohno’s relation and state new proofs for the  $\mathcal{S}$ -version of Theorem 3.3.9 and Theorem 3.3.12 as corollaries of Main Theorem.

We define a map  $\tau : \mathbb{R}[t] \rightarrow \mathbb{R}$  by  $\tau : t^n \mapsto \zeta(n)$  and its linear extension. We further extend  $\tau : \mathbb{R}[t][[x, y, z]] \rightarrow \mathbb{R}[[x, y, z]]$  coefficientwise. To show the corollaries, it is convenient to use the expression

$$\Delta_0^* \equiv \frac{1}{xy - z^2} \left( \exp \tau \left( \log \left( 1 + \frac{(xy - z^2)t^2}{1 + (x - y)t - xy t^2} \right) \right) - 1 \right) \pmod{\zeta(2)\mathcal{Z}}, \quad (4.3.1)$$

which follows from (4.2.2) and (4.2.3).

#### 4.3.1 Proof of Aoki–Ohno’s relation for SMZSVs

If  $x = y$ , then

$$\exp \tau \left( \log \left( 1 + \frac{(x^2 - z^2)t^2}{1 - x^2 t^2} \right) \right) \equiv 1 \pmod{\zeta(2)\mathcal{Z}}$$

because  $\tau(t^{2n}) \equiv 0$ . Thus

$$\Delta_0^*|_{x=y} \equiv \frac{1}{xy - z^2} \left( \exp \tau \left( \log \left( 1 + \frac{(xy - z^2)t^2}{1 + (x - y)t - xy t^2} \right) \right) - 1 \right) \Big|_{x=y} \equiv 0 \pmod{\zeta(2)\mathcal{Z}}. \quad (4.3.2)$$

On the other hand, by (4.1.11) we have

$$\begin{aligned} \Delta_0^*|_{x=y} &= \sum_{k,r,s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k,r,s)} \left( \zeta_{\mathcal{S}}^{*,*}(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) \right) x^{k-r-s} y^{r-s} z^{2s-2} \Big|_{x=y} \\ &= \sum_{k,s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k,*,s)} \left( \zeta_{\mathcal{S}}^{*,*}(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) \right) x^{k-2s} z^{2s-2}. \end{aligned} \quad (4.3.3)$$

Comparing (4.3.2) and (4.3.3), we obtain

$$\sum_{\mathbf{k} \in I_0(k, *, s)} \zeta_{\mathcal{S}}^*(\mathbf{k}) = \sum_{\mathbf{k} \in I_0(k, *, s)} \zeta^*(\mathbf{k}) \bmod \zeta(2)\mathcal{Z}.$$

By this relation and Aoki–Ohno’s relation for MZSVs (2.6.8), we obtain Corollary 4.1.2.

### 4.3.2 Proof of Sum formula for SMZSVs

Set

$$\begin{aligned} W &= \tau \left( \log \left( 1 + \frac{(xy - z^2)t^2}{1 + (x - y)t - xyt^2} \right) \right) \\ &= \tau \left( \frac{(xy - z^2)t^2}{1 + (x - y)t - xyt^2} - \frac{1}{2} \left( \frac{(xy - z^2)t^2}{1 + (x - y)t - xyt^2} \right)^2 + \dots \right). \end{aligned}$$

Since  $(xy - z^2) | W$ , we have

$$\Delta_0^* \Big|_{z^2=xy} = \frac{e^W - 1}{xy - z^2} \Big|_{z^2=xy} = \tau \left( \frac{t^2}{(1 + xt)(1 - yt)} \right).$$

By using the decomposition into partial fractions, we have

$$\begin{aligned} \frac{t^2}{(1 + xt)(1 - yt)} &= \frac{t}{x + y} \left( \frac{1}{1 - yt} - \frac{1}{1 + xt} \right) \\ &= \frac{t}{x + y} \sum_{m=0}^{\infty} (y^m - (-x)^m) t^m \\ &= \sum_{m=1}^{\infty} t^{m+1} \sum_{j=0}^{m-1} (-x)^{m-1-j} y^j \\ &= \sum_{j=0}^{\infty} \sum_{m=j+1}^{\infty} t^{m+1} (-x)^{m-1-j} y^j \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m t^{m+j+2} x^m y^j \\ &= \sum_{r=1}^{\infty} \sum_{k=r+1}^{\infty} (-1)^{k-r-1} t^k x^{k-r-1} y^{r-1}. \end{aligned}$$

Thus

$$\Delta_0^*|_{z^2=xy} = \sum_{k,r \geq 0} (-1)^{k-r-1} \zeta(k) x^{k-r-1} y^{r-1}.$$

On the other hand, by (4.1.11) we have

$$\begin{aligned} \Delta_0^*|_{z^2=xy} &= \sum_{k,r,s \geq 0} \left( \sum_{\mathbf{k} \in I_0(k,r,s)} \left( \zeta_S^{*,*}(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) \right) x^{k-r-s} y^{r-s} z^{2s-2} \Big|_{z^2=xy} \\ &= \sum_{k,r \geq 0} \left( \sum_{\mathbf{k} \in I_0(k,r,*)} \left( \zeta_S^{*,*}(\mathbf{k}) - \zeta^*(\mathbf{k}) \right) \right) x^{k-r-1} y^{r-1} \end{aligned}$$

and hence

$$\begin{aligned} \sum_{\mathbf{k} \in I_0(k,r,*)} \zeta_S^{*,*}(\mathbf{k}) - \sum_{\mathbf{k} \in I_0(k,r,*)} \zeta^*(\mathbf{k}) &\equiv (-1)^{k-r-1} \zeta(k) \pmod{\zeta(2)\mathcal{Z}} \\ &\equiv (-1)^r \zeta(k) \pmod{\zeta(2)\mathcal{Z}}. \end{aligned} \tag{4.3.4}$$

By (4.3.4) and the sum formula for MZSVs (2.6.7), we have Corollary 4.1.3.

### 4.3.3 Proof of Generalized height-one duality for SMZSVs

We rephrase Li's result mentioned in Theorem 2.6.7 in terms of  $\Phi_{0,S}^*(x, y, z)$ .

**Lemma 4.3.1.**

$$x\Phi_{0,S}^*(-x, y, z) - y\Phi_{0,S}^*(-y, x, z) \equiv 0 \pmod{\zeta(2)\mathcal{Z}}.$$

*Proof.* By using the reflection formula for the Gamma function,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we have

$$\begin{aligned} A(x, y) &\frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(x+a)\Gamma(x+b)}{\Gamma(x)\Gamma(y)} \\ &= -\frac{x-y}{ab} \frac{xyA(x, y)}{z^2(x-y)} \frac{\pi^2 ab}{\sin \pi a \sin \pi b} \frac{\Gamma(1+x+a)\Gamma(1+x+b)}{\Gamma(1+x)\Gamma(1+y)}. \end{aligned}$$

By (2.6.13), we have

$$\frac{\Gamma(1+x+a)\Gamma(1+x+b)}{\Gamma(1+x)\Gamma(1+y)} = \exp\left(\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} \left( (x+a)^n + (x+b)^n - x^n - y^n \right)\right).$$

By the Taylor expansion of the logarithm and (2.6.12),

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n \frac{t^n}{n} \left( (x+a)^n + (x+b)^n - x^n - y^n \right) &= \log \frac{(1+xt)(1+yt)}{(1+(x+a)t)(1+(x+b)t)} \\
&= \log \frac{(1+xt)(1+yt)}{1+(x+y)t - z^2 t^2} \\
&= -\log \frac{1+(x+y)t - z^2 t^2}{1+(x+y)t + xy t^2} \\
&= -\log \left( 1 - \frac{(xy+z^2)t^2}{(1+xt)(1+yt)} \right).
\end{aligned}$$

From this expression, we have

$$Z(x, y) = \frac{x-y}{xy+z^2} \left( \frac{xyA(x, y)}{z^2(x-y)} \frac{\pi^2 ab}{\sin \pi a \sin \pi b} \exp \tau \left( -\log \left( 1 - \frac{(xy+z^2)t^2}{(1+xt)(1+yt)} \right) \right) - 1 \right).$$

Since

$$\begin{aligned}
\frac{xyA(x, y)}{z^2(x-y)} &= \frac{\cos \pi(x+y) - \cos \pi \sqrt{(x+y)^2 + 4z^2}}{2\pi^2 z^2} \cdot \frac{\pi^2 xy}{\sin \pi x \sin \pi y} \cdot \frac{\sin \pi(x-y)}{\pi(x-y)} \\
&\equiv 1 \pmod{\zeta(2)\mathcal{Z}}, \\
\frac{\pi^2 ab}{\sin \pi a \sin \pi b} &\equiv 1 \pmod{\zeta(2)\mathcal{Z}},
\end{aligned}$$

we obtain

$$Z(x, y) \equiv \frac{x-y}{xy+z^2} \left( \exp \tau \left( -\log \left( 1 - \frac{(xy+z^2)t^2}{(1+xt)(1+yt)} \right) \right) - 1 \right) \pmod{\zeta(2)\mathcal{Z}}.$$

On the other hand, by (4.3.1), we have

$$x\Delta_0^*(-x, y, z) - y\Delta_0^*(-y, x, z) \equiv \frac{y-x}{xy+z^2} \left( \exp \tau \left( \log \left( 1 - \frac{(xy+z^2)t^2}{(1-xt)(1-yt)} \right) \right) - 1 \right) \pmod{\zeta(2)\mathcal{Z}}. \quad (4.3.5)$$

Therefore the sum of (2.6.11) and (4.3.5) is

$$\begin{aligned}
&x\Delta_0^*(-x, y, z) - y\Delta_0^*(-y, x, z) + x\Phi_0^*(-x, y, z) - y\Phi_0^*(-y, x, z) \\
&\equiv \frac{x-y}{xy+z^2} \left( \exp \tau \left( -\log \left( 1 - \frac{(xy+z^2)t^2}{(1+xt)(1+yt)} \right) \right) - \exp \tau \left( \log \left( 1 - \frac{(xy+z^2)(-t)^2}{(1-xt)(1-yt)} \right) \right) \right) \pmod{\zeta(2)\mathcal{Z}}.
\end{aligned}$$

Since  $\tau(t^{2n}) \equiv 0$ , we see

$$\tau\left(-\log\left(1 - \frac{(xy + z^2)t^2}{(1 + xt)(1 + yt)}\right)\right) \equiv \tau\left(\log\left(1 - \frac{(xy + z^2)(-t)^2}{(1 - xt)(1 - yt)}\right)\right) \pmod{\zeta(2)\mathcal{Z}},$$

which implies

$$x\Delta_0^*(-x, y, z) - y\Delta_0^*(-y, x, z) + x\Phi_0^*(-x, y, z) - y\Phi_0^*(-y, x, z) \equiv 0 \pmod{\zeta(2)\mathcal{Z}}.$$

By (4.1.11), we have

$$x\Phi_{0,S}^*(-x, y, z) - y\Phi_{0,S}^*(-y, x, z) \equiv 0 \pmod{\zeta(2)\mathcal{Z}}.$$

□

Lemma 4.3.1 immediately implies

$$\begin{aligned} & x\Phi_{0,S}^*(-x, y, z) - y\Phi_{0,S}^*(-y, x, z) \\ &= \sum_{m,n \geq s \geq 1} (-1)^s \left( (-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_S^{*,*}(\mathbf{k}) \right. \\ & \quad \left. - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_S^{*,*}(\mathbf{k}) \right) x^{m+1-s} y^{n+1-s} z^{2s-2}, \\ & \quad + \sum_{n \geq s \geq 1} (-1)^{n+s} \left( \sum_{\mathbf{k} \in I_0(n+s, s, s)} \zeta_S^{*,*}(\mathbf{k}) \right) (x^{n+1-s} - y^{n+1-s}) z^{2s-2}, \end{aligned}$$

and we have Corollary 4.1.4.

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